Efficiently solving special instances of NP-hard problems: Integer programs with bounded subdeterminants

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Integer linear programs

$$\begin{aligned} \text{maximize} \quad & p^\intercal x \\ \text{subject to} \quad & Ax \le b, \\ & x \in \mathbb{Z}^n, \end{aligned}$$

where $A \in \mathbb{Z}^{m \times n}$ is the constraint (coefficient) matrix, $b \in \mathbb{Z}^m$, $p \in \mathbb{Z}^n$.

Totally Δ-modular constraint matrices

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- totally unimodular (TU) if $\Delta(A) \leq 1$
- totally Δ-modular matrix (TΔ) if $\Delta(A) \leq \Delta$

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Can integer programs with TΔ constraint matrices be solved in polynomial time?

Totally Δ-modular constraint matrices

The conjecture is true in the following cases,

- $\Delta(A) = 1$,
- $\Delta(A) = 2$ (Artmann, Weismantel, Zenklusen '17),
- $\Delta(A) \leq \Delta$ and A has at most 2 nonzero entries per row (or per column) (Fiorini, Joret, Weltge, Y. '21)
- $\Delta(A) \leq \Delta$ and A is a network transposed matrix with a constant number of additional rows and columns (AFJKSWY '25)
- $\Delta(A) \leq \Delta$ and A has at most 2 nonzero entries per row (or per column) and a constant number of additional rows and columns (Kober '25)

Related results:

- randomized algorithm for strictly 3 and 4-modular constraint matrices (Nägele, Nöbel, Santiago, Zenklusen '24)
- deterministic algorithm for S-modular matrices for restricted set of polynomials S (Celaya, Kuhlmann, Weismantel '24)

Outline

- TΔ constraint matrices with at most 2 nonzero entries in each row.
- TΔ constraint matrices which is a transposed network matrix with k additional rows and columns.
- A specific problem: Total matching with TΔ constraint matrix.

TΔ constraint matrices with at most 2 nonzero entries in each row

Theorem (Fiorini, Joret, Weltge, Y. '21)

For every integer $\Delta \geq 0$ there exists a strongly polynomial-time algorithm for solving integer programs of the form

$$\begin{aligned} \textit{maximize} \quad & p^{\intercal} x \\ \textit{subject to} \quad & Ax \leq b, \\ & x \in \mathbb{Z}^n, \end{aligned}$$

where A is **TΔM** *and contains* **at most two nonzero entries in each row**.

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Theorem (Fiorini, Joret, Weltge, Y. '21)

For every integer $k \geq 0$ there exists a strongly polynomial-time algorithm for solving the maximum weight independent set problem in graphs G with at most k disjoint odd cycles.

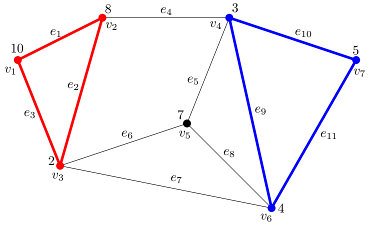
TΔ incidence matrix of a graph

Observation

Let G be a graph with k (vertex-) disjoint odd cycles. Then the constraint matrix in the maximum independent set problem contains a submatrix with determinant 2^k.

TΔ incidence matrix of a graph

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
e_1	1	1	0	0	0	0	0
e_2	0	1	1	0	0	0	0
e_3	1	0	1	0	0	0	0
e_4	0	1	0	1	0	0	0
e_5	0	0	0	1	1	0	0
e_6	0	0	1	0	1	0	0
e_7	0	0	1	0	0	1	0
e_8	0	0	0	0	1	1	0
e_9	0	0	0	1	0	1	0
e_{10}	0	0	0	1	0	0	1
e_{11}	0	0	0	0	0	1	1



The structure of graphs with bounded number of disjoint odd cycles

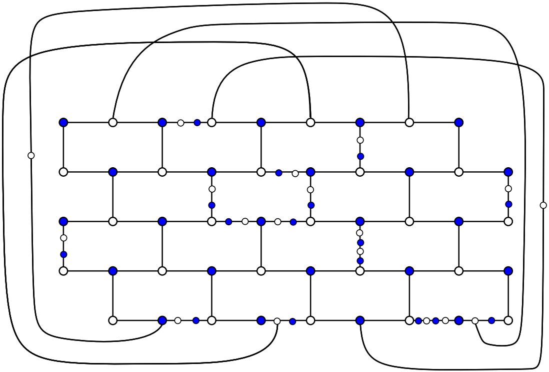
An easy case: There is a small set of vertices H such that $G \setminus H$ is bipartite.

The structure of graphs with bounded number of disjoint odd cycles

Otherwise, G has a bounded number of disjoint odd cycles and no set H whose removal makes the graph bipartite.

The structure of graphs with bounded number of disjoint odd cycles

Escher wall:



G contains a large Escher wall (Reed '99).

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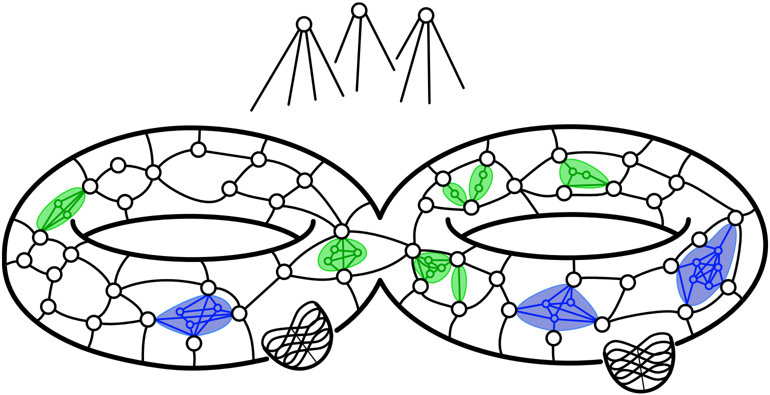
Using the version of the Excluded Minor Structure Theorem of Robertson and Seymour due to Kawarabayashi, Thomas, and Wollan '20 we obtain a structural characterization.

The structure of graphs with bounded number of disjoint odd cycles



After removing a bounded number of vertices, a subgraph G_0 of G can be embedded on a bounded genus surface $\mathbb{S}$, where the rest of G is partitioned between a bounded number of *"large vortices"* and possibly unbounded number of *"small vortices"*.

TΔ constraint matrices which is a transposed network matrix with k additional rows and columns

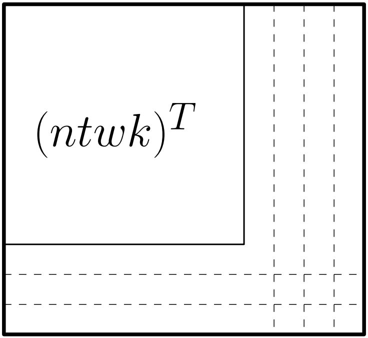
Theorem (Aprile, Fiorini, Joret, Kober, Seweryn, Weltge, Y. '25)

Let $\Delta \in \mathbb{N}$, then there exists a strongly polynomial-time algorithm for solving integer programs of the form

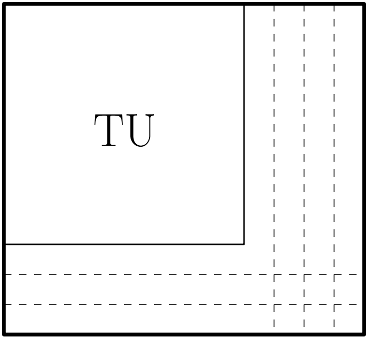
$$\begin{array}{ll} \textit{maximize} & p^T x \\ \textit{subject to} & Ax \leq b \\ & x \in \mathbb{Z}^n, \end{array}$$

where A is **TΔM** *and* **is a transposed network matrix with $\leq k$ additional rows and columns**.

TΔ constraint matrices which is a transposed network matrix with k additional rows and columns



Why this case?



Why this case?

Theorem (Seymour '80, informal)

Let A be a TU matrix then one of the following holds,

- *A is a network matrix,*
- *A is transposed network matrix (denoted $(ntwk)^T$),*
- *A is one of two well-defined 5×5 matrices,*
- *A can be obtained through "simple block operations" (1-sum, 2-sum, 3-sum) applied on smaller TU matrices.*

Why this case?

1. partially ordered knapsack

$$\max\{c^\intercal x : x_i \le x_j \ \forall i \preceq j,\ w^\intercal x \le d,\ x \in \{0,1\}^n\}$$

2. densest k-subgraph in a graph $G = (V, E)$

$$\max\left\{ \sum_{e \in E} x(e) : \begin{array}{l} x(e) \le x(v) \ \forall v \in V,\ \forall e \in \delta(v), \\ \sum_{v \in V} x(v) = k,\ x \in \{0,1\}^{V \cup E} \end{array} \right\}$$

A simplified version

Theorem (Aprile, Fiorini, Joret, Kober, Seweryn, Weltge, Y. '25)

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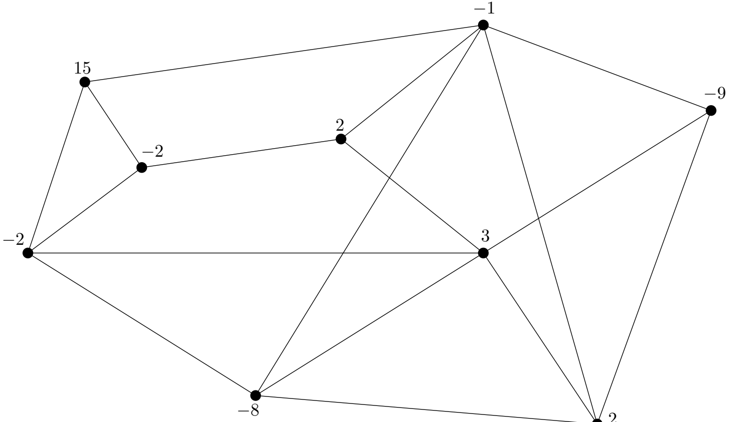
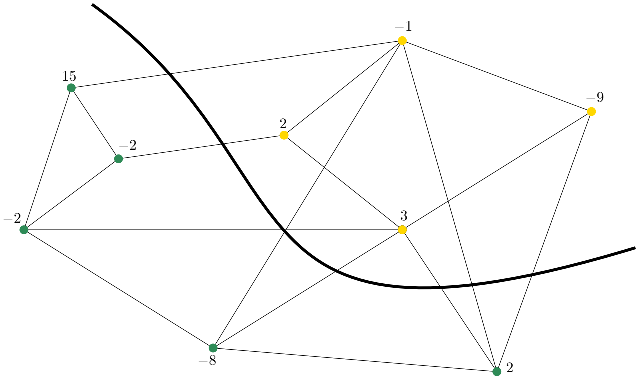
where A is **TΔM** *and* **is a transposed network matrix with 1 additional row**.

Totally Δ-modular (ntwk)$^{\mathsf{T}}$ + one row



Docsets

A set $S \subseteq V$ is doubly connected (in short, a docset) if $G[S]$ and $G[V \setminus S]$ are connected.



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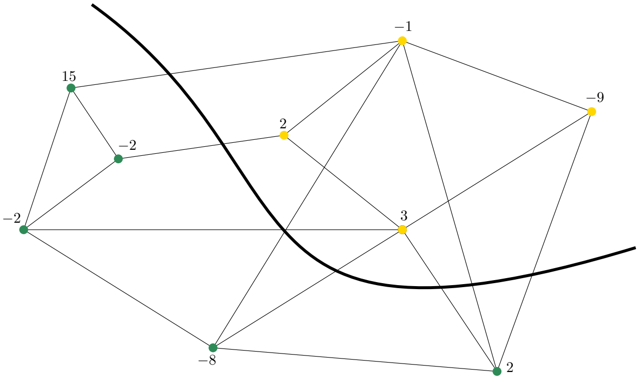

The weight of a docset is the sum of the weights of the vertices.

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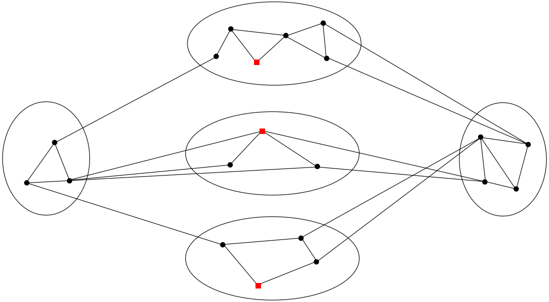
Lemma

$$\Delta = \max_{S \text{ is a docset}} |weight(S)|.$$

Rooted graphs

Lemma

Let $\Delta \in \mathbb{N}$ and let G be a graph with no docset of weight larger than Δ, then G does not contain a rooted $K_{f(\Delta)}$-model.



Embedded graphs without a rooted $K_{2,t}$-minor

Theorem (Böhme, Mohar '02, Fiorini, Kober, Seweryn, Shantanam, Y. '25+)

Let G be a 3-connected planar graph without a rooted $K_{2,t}$-minor, then all the roots are contained in $f(t)$ faces.

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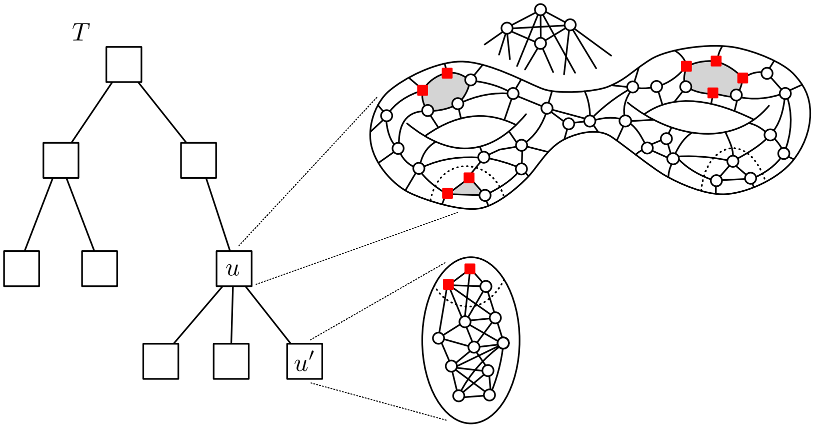
Theorem (Böhme, Kawarabayashi, Maharry, Mohar '08, Fiorini, Kober, Seweryn, Shantanam, Y. '25+)

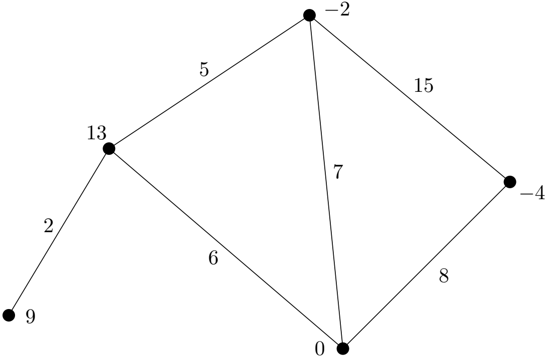
Let G be a 3-connected graph embedded on a surface of genus g with large face-width and without a rooted $K_{2,t}$-minor, then all the roots are contained in $f(t, g)$ faces.

The structure of graphs without rooted $K_{2,t}$-minors



Total matching

Problem: Let G be a graph with weights on vertices and edges. Find a maximum weight set of elements $X \subseteq V(G) \cup E(G)$ such that no two elements of X are adjacent or incident.



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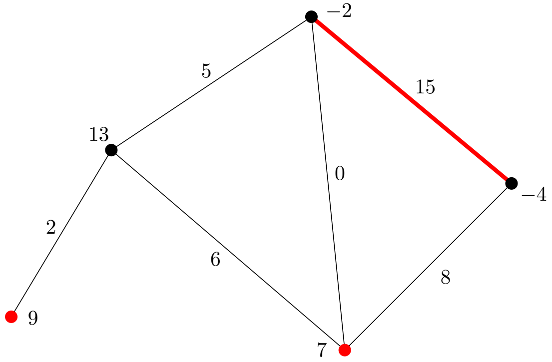

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$$\begin{array}{rl} \max & \sum_{v \in V(G)} p(v) x_v + \sum_{e \in E(G)} p(e) y_e \\ \text{s.t.} & x_v + \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V(G) \\ & x_v + x_w + y_e \leq 1 \quad \forall e = vw \in E(G) \\ & x \in \{0,1\}^n, \ y \in \{0,1\}^m \end{array}$$

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Theorem (Ferrarini, Fiorini, Kober, Y. '24)

Let G be a graph for which the constraint matrix in the above formulation is TΔM, then there is an algorithm which finds the maximum weight total matching in G in polynomial time.

Total matching

Observation

A cycle has a determinant at least 2.

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The max degree in G *is* $\Delta + 1$.

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Claim

There are $O(\log \Delta)$ *vertices of degree at least* 3 *in* G.

Future directions

- Polynomial time algorithm for the case of ntwk + constant number of rows?

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Thank you.