Sparse Integer Solutions and Approximations

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What is Sparsity?

Consider a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$ such that

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}.$$

Suppose $|\text{supp}(\boldsymbol{x})| =$ "#nonzero entries" is large.

Does there exist $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{y}$ with small $|\text{supp}(\boldsymbol{y})|$?

Why you might care?

- In practice, sparse solutions are preferred.
- Sparse solutions often give fast(er) algorithms: Reduces problems with many variables to subproblems with fewer variables.

What is Known?

Consider a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$ such that

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In terms of m and $\|\boldsymbol{A}\|_\infty$:

$$|\text{supp}(\boldsymbol{y})| \lesssim \mathcal{O}(m \cdot \log_2(c \cdot \sqrt{m} \cdot \|\boldsymbol{A}\|_\infty))$$

(Eisenbrand, Shmonin '06; Aliev, De Loera, Oertel, O'Neill '17; Berndt, Jansen, Klein '21,...)

This tight up to the constant c!

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Does there exist $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{y}$ with small $|\text{supp}(\boldsymbol{y})|$?

In terms of determinants of $\boldsymbol{A}$:

$$|\text{supp}(\boldsymbol{y})| \leq m + \log_2(\sqrt{\det \boldsymbol{A}\boldsymbol{A}^\top})$$

(Aliev, Averkov, De Loera, Oertel, O'Neill '17 & '21; Lee, Paat, Stallknecht, Xu '20; Gribanov, Shumilov, Malyshev, Pardalos '24,...)

What is the Plan for Today?

1. Discuss a novel bound in terms of Δ, the largest subdeterminant $\boldsymbol{A}$:

$$|\text{supp}(\boldsymbol{y})| \leq m + \mathcal{O}(\Delta^2).$$

   (ongoing work with Robert Weismantel)

2. Discuss a more general problem: for a fixed $k < n$, can we find $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ with $|\text{supp}(\boldsymbol{y})| \leq k$ such that

$$\boldsymbol{b} \approx \boldsymbol{A}\boldsymbol{y}?$$

   (jointly with Timm Oertel and Robert Weismantel (IPCO 2025))

1. A Novel Upper Bound

Folklore

Linear programs: Finding an optimal solution $\boldsymbol{x}^*$ to a linear program with

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}$$

can be done efficiently (polynomial time) and

$$|\text{supp}(\boldsymbol{x}^*)| \leq m.$$

Integer linear programs: Finding an optimal solution $\boldsymbol{z}^*$ to the corresponding integer linear program is difficult (NP-hard) and $\boldsymbol{z}^*$ has more complicated structure than $\boldsymbol{x}^*$.

A Novel Upper Bound

Let Δ be the largest full rank subdeterminant of $\boldsymbol{A}$.

Theorem (jointly with Weismantel)

For each (feasible) integer linear program defined by $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$*, there exists an optimal solution* $\boldsymbol{z}^*$ *such that*

$$|\text{supp}(\boldsymbol{z}^*)| \leq m + f(\Delta),$$

where $\Delta - 1 \leq f(\Delta) \leq \left\lceil \frac{\Delta-1}{2} \right\rceil \cdot (\Delta - 1)$.

1. $f(1) = 0$: $|\text{supp}(\boldsymbol{z}^*)| \leq m$ (Hoffmann, Kruskal '56),
2. $f(2) = 1$: $|\text{supp}(\boldsymbol{z}^*)| \leq m + 1$ (Veselov, Chirkov '09),
3. $f(3) = 2$: $|\text{supp}(\boldsymbol{z}^*)| \leq m + 2$ (new!).

In general:

$$\boldsymbol{z}^* \text{ lies on a face of dimension at most } f(\Delta) \text{ of } \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}.$$

What is the New Idea?

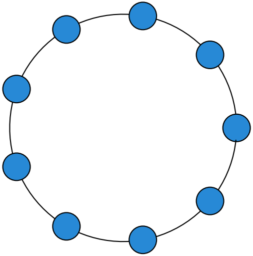
- Goal: Let Δ be the largest full rank subdeterminant of $\boldsymbol{A}$, then

$$|\text{supp}(\boldsymbol{z}^*)| \leq m + \left\lceil \frac{\Delta - 1}{2} \right\rceil \cdot (\Delta - 1).$$

- Already known approach: Find $\boldsymbol{y} \in \{-1, 0, 1\}^n$ s.t. $\boldsymbol{Ay} = \boldsymbol{0}$.
- In general, $\boldsymbol{y}$ does not exist.

$\boldsymbol{y}$ exists when the number of variables is large and Δ is fixed!

$$\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{pmatrix}$$



2. Beyond Sparse Exact Solutions

Beyond Sparsity

Consider a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$.

Fix $k < n$. Does there exist $y \in \mathbb{Z}^n_{\geq 0}$ with $|\text{supp}(y)| \leq k$ such that

$$\boldsymbol{b} \approx \boldsymbol{A}\boldsymbol{y}?$$

Goal:

- Understand the trade-off between sparsity and feasibility.

Our key finding:

Approximation error decreases exponentially as $k \to n$.

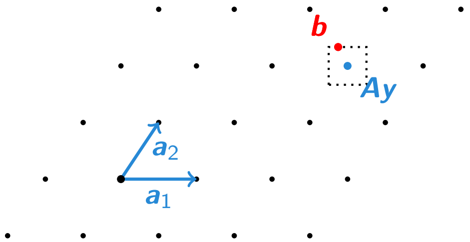
How This May Look: Lattices

Theorem (jointly with Oertel and Weismantel)

Given $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $\boldsymbol{x} \in \mathbb{Z}^n$, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$, *there exist* $\boldsymbol{y} \in \mathbb{Z}^n$ *with* $|\text{supp}(\boldsymbol{y})| \leq k$ *such that*

$$\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{y}\|_\infty \leq \frac{1}{2^{k-m+1}} \cdot \delta(\boldsymbol{A})$$

Proof sketch:



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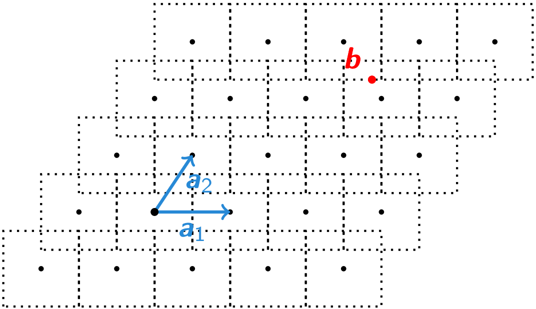

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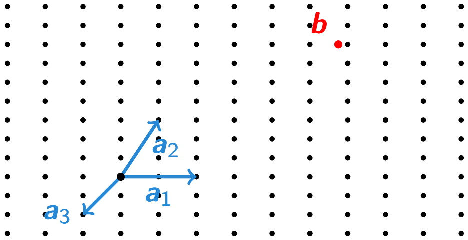

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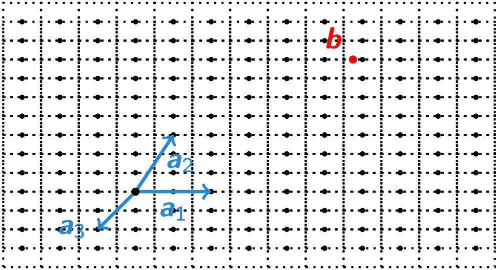
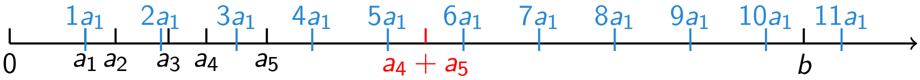
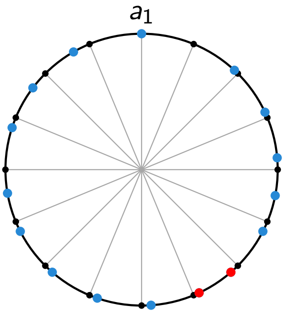


$m = 1$: From n to $n - 1$

Let $\boldsymbol{a}^\top = (a_1, \ldots, a_n)$, $\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$, $|\text{supp}(\boldsymbol{x})| = n$, and $b = \boldsymbol{a}^\top \boldsymbol{x}$.



Consider $b = a_1 + \ldots + a_n$. There are 2^{n-1} subsums (ignore a_1).



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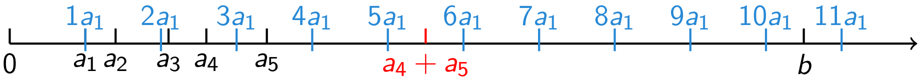


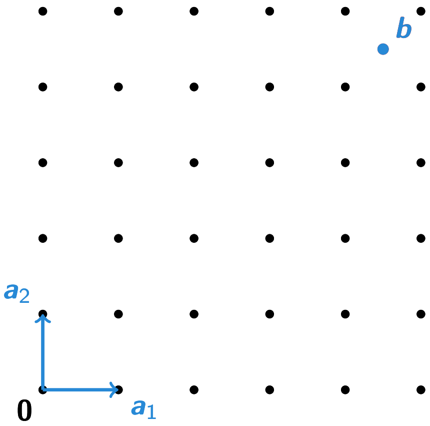
Theorem (jointly with Oertel and Weismantel)

There exists $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ *with* $|\mathrm{supp}(\boldsymbol{y})| \leq k$ *such that*

$$\left|\boldsymbol{a}^\top \boldsymbol{y} - b\right| \leq \left(\frac{1}{2^{k-1}} - \frac{1}{2^{n-1}}\right) \cdot a_1 .$$

This bound is tight when $k = n - 1$.

$m \geq 2$: From n to $n-1$



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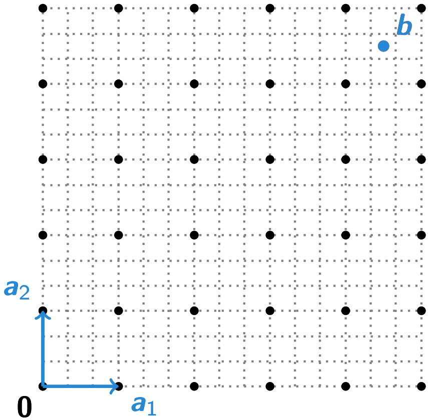

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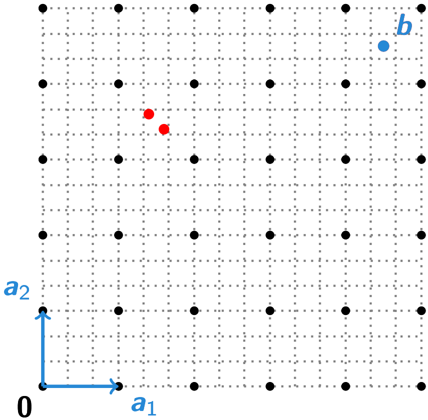

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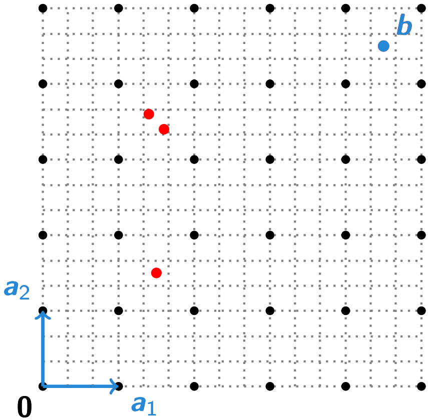

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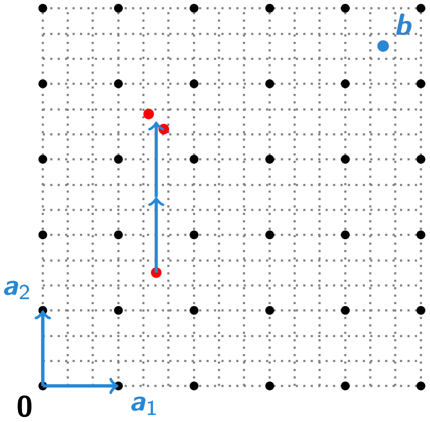

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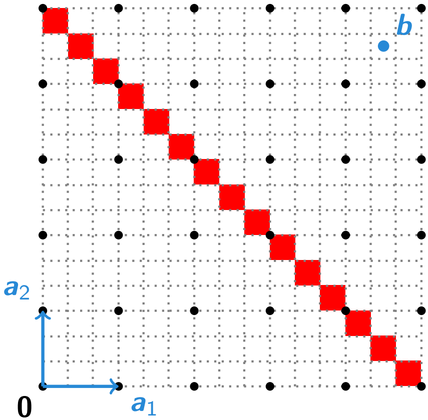

Number of **incomparable** boxes $\approx \text{size}(\boldsymbol{b})^{m-1}$.

$m \geq 2$

Let $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$. Fix $\boldsymbol{B} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$ with $\boldsymbol{a}_i \in \operatorname{pos}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

Theorem (jointly with Oertel and Weismantel)

There exists $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ with $|\operatorname{supp}(\boldsymbol{y})| \leq k$ such that

$$\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{y}\|_{P(\boldsymbol{B})} \leq \frac{1}{2^{\frac{1}{m}} - 1} \cdot \left(\frac{1}{2^{\frac{k-m}{m}}} - \frac{1}{2^{\frac{n-m}{m}}} \right) \cdot size(\boldsymbol{b})^{\frac{m-1}{m}}$$

Important step:

It suffices to consider only finitely many b!

From this, we get

$$\operatorname{size}(\boldsymbol{b}) \leq \max \|\boldsymbol{a}_i\|_{P(\boldsymbol{B})} \cdot |\det \boldsymbol{B}| \,.$$

Next Steps

1. Better bounds when k is small, e.g., $k = \mathcal{O}(m)$ or even $k = 3, m = 1$.

Theorem (jointly with Oertel and Weismantel)

There exists $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ *with* $|\text{supp}(\boldsymbol{y})| \leq 2$ *such that*

$$\left| \boldsymbol{a}^\top \boldsymbol{y} - b \right| \leq 0.2901\ldots \cdot a_1.$$

This bound is tight.

2. What is possible to compute in polynomial time, where k is part of the input?

Thanks for your attention!