Semidefinite approximations for biindependent pairs and bicliques in bipartite graphs

Monique Laurent

CWI

Tilburg University

Joint work with **Sven Polak** (Tilburg University)

and **Luis Felipe Vargas** (IDSIA Lugano)

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Define the **balanced parameters** $\alpha_{\text{bal}}(G)$, $g_{\text{bal}}(G)$, $h_{\text{bal}}(G)$, where we optimize over **balanced biindependent** pairs.

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Definition (biindependent pairs and bicliques in general graphs)

Let $G = (V, E)$ be a (general) graph and let $A, B \subseteq V$ be disjoint.

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Lemma (Links to bipartite graphs)

Let $G = (V, E)$ be a (general) graph.

- *The* **(extended) bipartite double** $B_0(G)$ *of G has bipartition $V \cup V'$, and edges $\{i, j'\}, \{j, i'\}$ for $\{i, j\} \in E$, and $\{i, i'\}$ for $i \in V$.*

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- *A pair (A, B) of disjoint subsets of V is biindependent in G*

  $\iff (A, B')$ *is biindependent in the bipartite graph $B_0(G)$.*

Outline

- Motivation
- Relations between the bounds
- Semidefinite and eigenvalue bounds
- Relations between the semidefinite bounds
- Relations to other eigenvalue bounds
- Examples

Motivation

Some applications

- Computing the **maximum number of nodes in a balanced biclique** has applications, e.g., to VLSI design [Al-Yamani et al.'07, Tohhory'06], for analysis of biological data [Yang et al.'05], protein interactions modeling (Mukhopadhyay et al.'14]

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  This connection was communicated to us by Frank Vallentin (2020) and motivated our research.

  A few details in next slide.

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we get $\boxed{\varphi(\Gamma) \leq \frac{|\Gamma|}{1+k^{1/3}}}$ sharpening Gower's bound $\boxed{\varphi(\Gamma) \leq \frac{|\Gamma|}{k^{1/3}}}$

Relations between the (balanced) parameters

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Moreover, $\alpha(G) = \alpha_{\text{bal}}(G) \iff h(G) = \frac{1}{4}\alpha(G) \iff \frac{1}{2}\sqrt{g(G)} = \frac{1}{4}\alpha(G)$.

- The right most inequalities are based on $|A| + |B| \ge 2\sqrt{|A|\cdot|B|}$:

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- The non-trivial implication follows from the fact that equality $|A| + |B| = 2\sqrt{|A| \cdot |B|}$ holds iff $|A| = |B|$.

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Theorem (L-Polak-Vargas, MOR 2025)

For a bipartite graph G, testing whether $\alpha(G) = \alpha_{\text{bal}}(G)$ is an **NP-complete** *problem.*

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For a bipartite graph G, testing whether $\alpha(G) = \alpha_{\text{bal}}(G)$ is an **NP-complete** *problem.*

Hence, it is **NP-hard** *to compute $h(G)$ and $g(G)$.*

Semidefinite and eigenvalue bounds

Semidefinite bounds

$\boxed{h(G) \leq \frac{1}{2}\sqrt{g(G)} \leq \frac{1}{4}\alpha(G)}$ for $G = (V_1 \cup V_2, E)$ bipartite graph

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Pf: (A, B) biindependent $\rightsquigarrow x = \chi^{A \cup B} \rightsquigarrow X = xx^{\mathsf{T}}$

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Compare with **Lovász theta number** (for any G):

$$\vartheta(G) = \max_{X \in \mathcal{S}^V} \{\langle J, X \rangle : X \succeq 0,\ \mathrm{Tr}(X) = 1,\ X_{ij} = 0\ (\{i,j\} \in E\})$$

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Recall: $\alpha(G) \le \vartheta(G) = \mathrm{las}_1(G)$, with equality if G is bipartite

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Note: Replace C by J in h_1 (resp., by I in g_1) $\rightsquigarrow$ get ϑ (resp., las_1).

Relations between the semidefinite bounds

$$\boxed{h(G) \leq \frac{1}{2}\sqrt{g(G)} \leq \frac{1}{4}\alpha(G)} \quad \boxed{h(G) \leq h_1(G)} \quad \boxed{g(G) \leq g_1(G)}$$

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For any bipartite graph G

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(see later)

About the proofs of $\frac{1}{2}\sqrt{g(G)} \le h_1(G) \le \frac{1}{2}\sqrt{g_1(G)}$

- For the inequality $\boxed{\frac{1}{2}\sqrt{g(G)} \le h_1(G)}$

Use the **dual semidefinite formulation** of $h_1(G)$:

$$\boxed{h_1(G) = \min_{\lambda \in \mathbb{R}, Z \in \mathcal{S}^V} \{\lambda : \lambda I + Z - C \succeq 0,\ Z_{ij} = 0\ (i = j, \{i,j\} \in \overline{E})\}}$$

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and show that $\lambda \geq \frac{1}{2}\sqrt{|A| \cdot |B|}$ if (A, B) is a biindependent pair,

by considering the principal submatrix of $\lambda I + Z - C$ indexed by $A \cup B$.

About the proofs of $\frac{1}{2}\sqrt{g(G)} \leq h_1(G) \leq \frac{1}{2}\sqrt{g_1(G)}$

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• For the inequality $\boxed{h_1(G) \leq \frac{1}{2}\sqrt{g_1(G)}}$, the proof is somewhat analog to the classical proof for showing $\boxed{\vartheta(G) \leq \mathrm{las}_1(G)}$, but more involved

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- In comparison, the **dual formulation** of $g_1(G)$ reads:

$$g_1(G) = \min_{\lambda, u\in\mathbb{R}^V, Z\in\mathcal{S}^V} \{\lambda : \begin{pmatrix} \lambda & u^\mathsf{T}/2 \\ u/2 & \mathrm{Diag}(u) - C + Z \end{pmatrix} \succeq 0,\ Z_{ij} = 0\ (i = j, \{i,j\} \in \overline{E})\}$$

Eigenvalue bounds

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Definition (Eigenvalue bounds $h_{\mathrm{eig}}(G)$ and $g_{\mathrm{eig}}(G)$)

Assume G is **bipartite** r**-regular**, $|V_1| = |V_2| = n$, and let λ_2 be the second largest eigenvalue of the adjacency matrix A_G. Then,

$$h_1(G) \leq h_{\mathrm{eig}}(G) = \frac{n\lambda_2}{2(\lambda_2+r)}, \qquad ,$$

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Key proof idea: Use the dual semidefinite programs and restrict to symmetric solutions: $Z = tA_G$, $u = se$

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Lemma (Relation between the eigenvalue bounds)

We have: $h_{\mathrm{eig}}(G) \le \frac{1}{2}\sqrt{g_{\mathrm{eig}}(G)}$

Relation to Haemers spectral bound

Haemers (2003) studied bicliques and biindependent pairs in general graphs.

Proposition (Link to a bound of Haemers (2003))

Let $G = (V, E)$ a graph, $n = |V|$, and $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ the eigenvalues of the Laplacian of G. Let $B_0(G)$ be the **extended bipartite double** *of G.*

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Moreover, equality $h_1(B_0(G)) = \frac{n}{4}\left(1 - \frac{\mu_2}{\mu_n}\right)$ holds if G is vertex- and edge-transitive.

Recall: $g(B_0(G))$ is the maximum edge-cardinality of a biclique in $\overline{G}$.

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• However, all natural symmetrizations of these bounds **do not improve** the spectral parameter $h_{\mathrm{eig}}(G)$.

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Q_r has vertex set $V = \{0,1\}^r$, with an edge $\{u,v\}$ if $d_H(u,v) = 1$.

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We show that this is **true** for $r \leq 13$ (using higher order Lasserre bound)

Conclusions

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- Open problem: Compute $\alpha_{\mathrm{bal}}(Q_r)$ for the Hamming cube, and verify whether it is given by the sequence $a(r-1)$.

THANK YOU !