A Characterization of Unimodular Hypergraphs with Disjoint Hyperedges

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Totally unimodular matrices

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Truemper's characterization of almost TU matrices (Truemper '92):
recursive description of the class of almost TU (i.e., minimally non-TU) matrices

Incidence matrices of graphs

A **graph** is a pair $G = (V, E)$ of

- a finite set V of **vertices** and
- a finite multiset $E \subseteq \binom{V}{2}$ of **edges**.

The **incidence matrix** $\boldsymbol{M}(G) \in \{0,1\}^{V \times E}$ of G is defined by $\boldsymbol{M}(G)_{v,e} \begin{cases} 1, & v \in e \\ 0, & v \notin e \end{cases}$.

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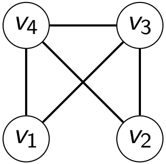

We call G **unimodular** if $\boldsymbol{M}(G)$ is TU.

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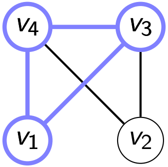

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Theorem (Grossman, Kulkarni, and Schochetman '95)

For a graph G with odd cycle packing number $ocp(G)$, we have $\Delta(\boldsymbol{M}(G)) = 2^{ocp(G)}$.
In particular, G is unimodular if and only if G is bipartite/ contains no odd cycle.

odd cycle packing number: maximum number of pairwise vertex-disjoint odd cycles

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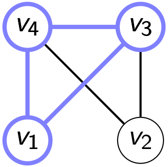

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For a graph G with odd cycle packing number ocp(G), we have $\Delta(\boldsymbol{M}(G)) = 2^{ocp(G)}$.
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(generalizes to mixed graphs $\triangleq$ $\{0, \pm 1\}$-matrices with ≤ 2 non-zeros per column)

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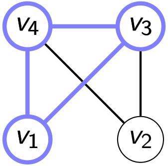

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What about **hypergraphs**?

Incidence matrices of hypergraphs

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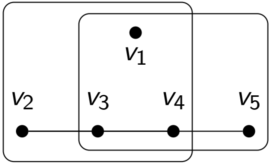

$e = \{v_1, v_2, v_3, v_4\}$

$f = \{v_1, v_3, v_4, v_5\}$

We call G **unimodular** if $M(G)$ is TU.

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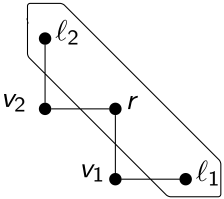
General (directed) hypergraphs correspond to arbitrary $\{0,1\}$-($\{0,\pm 1\}$)-matrices.

Research question:
Can we obtain a **simple characterization** of unimodularity via **forbidden subhypergraphs** for a **special class** of hypergraphs?

Partial subhypergraphs

A **partial subhypergraph** $H \subseteq_P G$ has the form $H = G[U, F] := (U, F[U])$, where $U \subseteq V$, $F \subseteq E$, and $F[U] := \{f \cap U : f \in F\}$.



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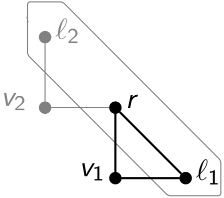

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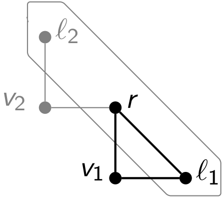

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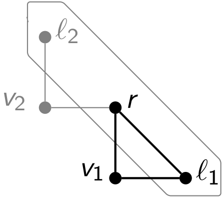

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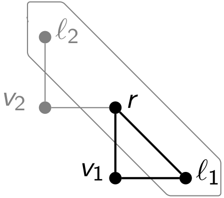

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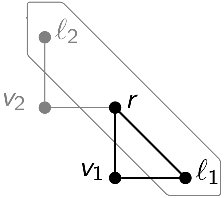

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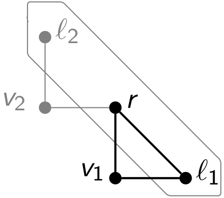

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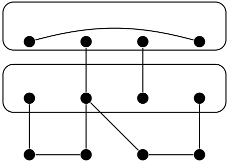
Camion's characterization of TU matrices (phrased in terms of hypergraphs)

A hypergraph is **Eulerian** if every vertex has even degree and every hyperedge has even size.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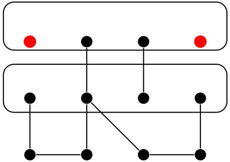

NOT Eulerian

Theorem (Camion '65)

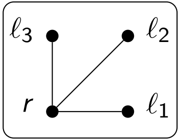
A hypergraph G is unimodular if and only if for every Eulerian $H \subseteq_P G$, $|\text{supp}(\boldsymbol{M}(H))|$ is divisible by 4, where $\text{supp}(\boldsymbol{M}(H)) := \{(v,e) \colon \boldsymbol{M}(H)_{v,e} = 1\}$.

In particular: G is unimodular $\Leftrightarrow$ every Eulerian $H \subseteq_P G$ is unimodular

If every hyperedge in $E(G)$ has size ≤ 3:
G is unimodular $\Leftrightarrow$ every graph $H \subseteq_P G$ is unimodular $\Leftrightarrow$ G contains no odd cycle

Larger hyperedges

If G contains hyperedges of size ≥ 4, forbidding odd cycles is not sufficient for unimodularity[1]:



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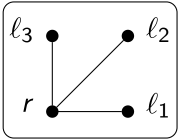

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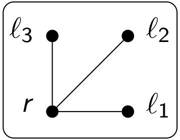

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Our result: For disjoint hypergraphs, forbidding odd cycles and partial subhypergraphs "similar to the one above" is sufficient to guarantee unimodularity.

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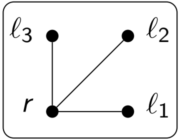

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$\underbrace{\text{“similar to the one above”}}_{\text{we call them odd tree houses}}$

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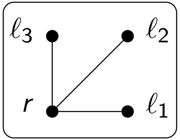
Odd tree houses

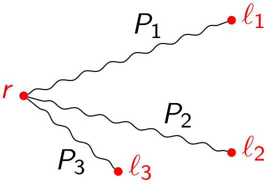
An **odd tree house** is a hypergraph consisting of the following:

- a hyperedge $e = \{r, \ell_1, \ell_2, \ell_3\}$ of size 4
- an odd r-ℓ_i-path P_i for $i \in [3]$

The paths $(P_i)_{i\in[3]}$ are pairwise edge-disjoint and do not share any vertex other than r.



$$\begin{array}{c|cccc} & \{r,\ell_1\} & \{r,\ell_2\} & \{r,\ell_3\} & \{r,\ell_1,\ell_2,\ell_3\} \\ \hline r & 1 & 1 & 1 & 1 \\ \ell_1 & 1 & 0 & 0 & 1 \\ \ell_2 & 0 & 1 & 0 & 1 \\ \ell_3 & 0 & 0 & 1 & 1 \end{array}$$

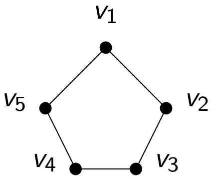


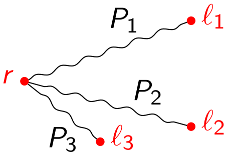
Observation: Let T be an odd tree house. $\Delta(\boldsymbol{M}(T)) = 2$ and $\boldsymbol{M}(T)$ is almost TU.

Our main result

Theorem (Caoduro, N., Paat 25+)

Let G be a disjoint hypergraph. Then G is unimodular if and only if there does not exist $H \subseteq_P G$ that is an odd cycle or an odd tree house.





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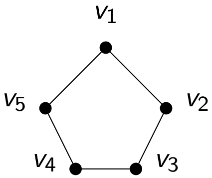
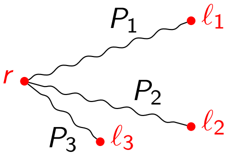
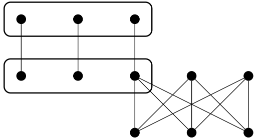




Incidence matrices of unimodular disjoint hypergraphs are, in general, neither network nor co-network matrices.



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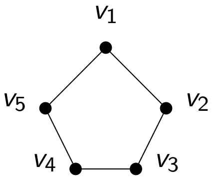
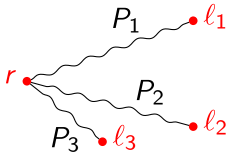
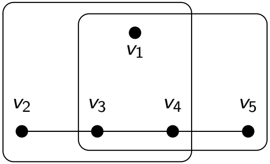




The result does not generalize to non-disjoint hypergraphs. (example from (Cornuéjols & Zuluaga '00))



$e = \{v_1, v_2, v_3, v_4\}$

$f = \{v_1, v_3, v_4, v_5\}$

	e	f	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_4, v_5\}$
v_1	1	1	0	0	0
v_2	1	0	1	0	0
v_3	1	1	1	1	0
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v_5	0	1	0	0	1

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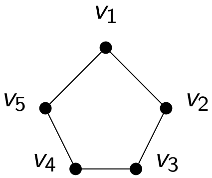

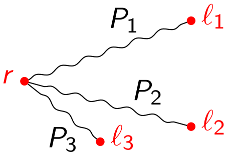

The result naturally extends to disjoint directed hypergraphs (more details later).

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Examples of problems that can be modeled as an IP whose constraint matrix is the incidence matrix of a disjoint hypergraph:

- fixed-cardinality independent set

fixed-cardinality independent set: Given a graph $G = (V, E)$, $w\colon V \to \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$, find an independent set in G of size k of minimum weight or decide that none exists.

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- fixed-cardinality independent set
- job interval selection

job interval selection: Given sets of intervals $(\mathcal{I}_i)_{i=1}^n$, select $I_i \in \mathcal{I}_i$ ("interval for job i") such that no two of the intervals $I_1, \ldots, I_n$ overlap, or decide that's not possible.

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- fixed-cardinality independent set
- job interval selection
- fair representation by independent sets

fair representation by independent sets: Given a graph $G = (V, E)$, a partition $V = V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_k$ and $\alpha \geq 0$, decide whether there exists an independent set S with $|S \cap V_i| \geq \alpha \cdot |V_i|$ for $i \in [k]$.

Our main result

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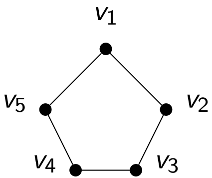
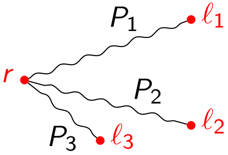
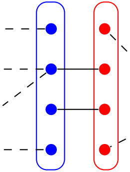




Proof idea: Removing even cycles

Assume G^* is a disjoint hypergraph that contains no odd cycle or tree house, but is not unimodular.



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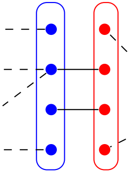


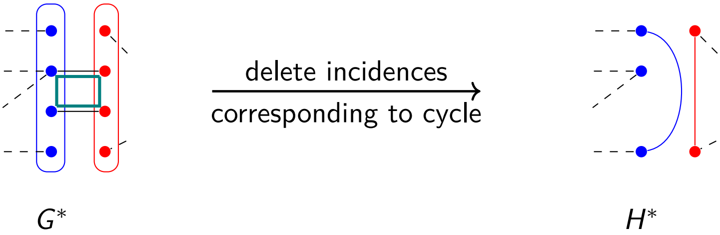
G^*

$\rightarrow$ Among all counterexamples, suppose G^* minimizes $|V(G^*)| + |E(G^*)|$.

$\rightarrow$ By Camion's theorem, G^* is Eulerian and $|\text{supp}(\boldsymbol{M}(G^*))| \not\equiv 0 \mod 4$.

Proof idea: Removing even cycles

We try to make G^* smaller by 'removing' an even cycle to obtain H^*.



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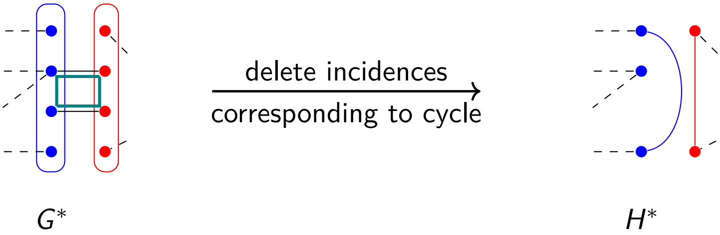

Then H^* is Eulerian, $|\text{supp}(\boldsymbol{M}(H^*))| \not\equiv 0 \mod 4$ and $|V(H^*)| + |E(H^*)| < |V(G^*)| + |E(G^*)|$.

Goal: Show that H^* contains no odd cycle and no odd tree house.

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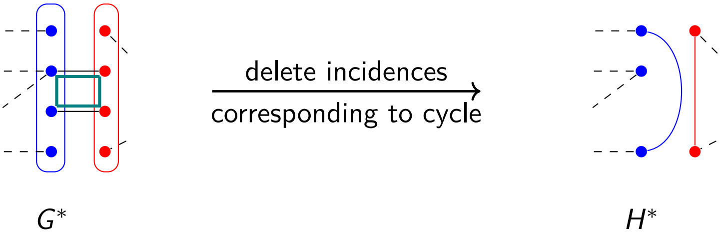

Problem: H^* is **not** necessarily a **partial subhypergraph** of G^*,

but only what we call a **quasi-subhypergraph** of G^*.

$\rightarrow$ odd cycle/ odd tree house in $H^* \not\Rightarrow$ odd cycle/odd tree house in G^*

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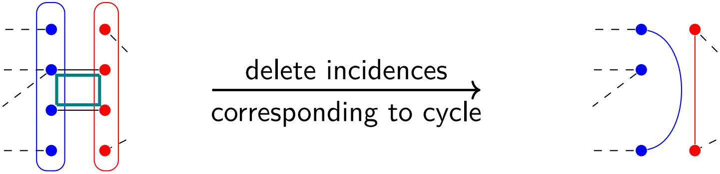


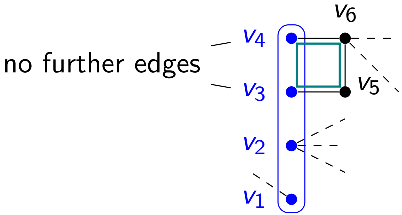
G^* H^*

Solution: Remove a very structured even cycle that 'brings H^* as close to being a partial subhypergraph of G^* as possible'

Analyze the structure of **conflicts** to prove that if H^* contains an odd cycle/ odd tree house, then so does G^*.

Nice cycles

Imagine we could find an even cycle $C = G^*[U, F] \subseteq_P G^*$ with the following property: For each hyperedge $e \in E_{\geq 4}(G^*) \cap F$ and each $v \in e \cap U$, $\delta_{G^*}(v) \subseteq F$.



	e	$\{v_3, v_5\}$	$\{v_5, v_6\}$	$\{v_4, v_6\}$							
v_1	1	0	0	0	1	0	0	0	0	0	...
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v_3	1	1	0	0	0	0	0	0	0	0	...
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v_5	0	1	1	0	0	0	0	0	0	0	...
v_6	0	0	1	1	0	0	0	0	1	1	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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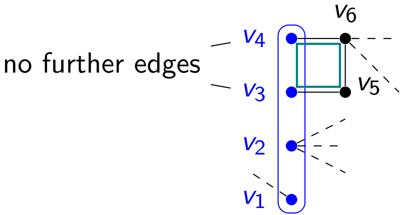

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v_6	0	0	1	1	0	0	0	0	1	1	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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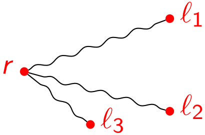
Problem: such a cycle does not always exist



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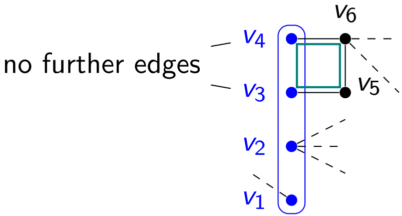

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	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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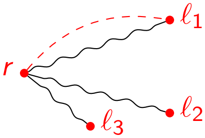

Finding a nice cycle

Properties of minimal counterexample G^*:

- G^* is Eulerian.
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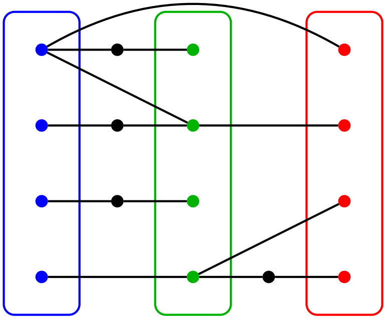

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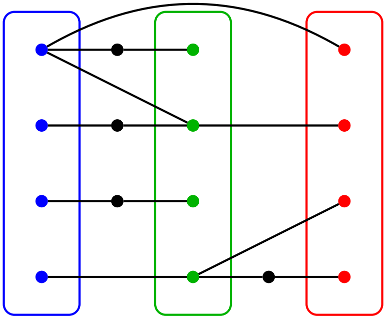

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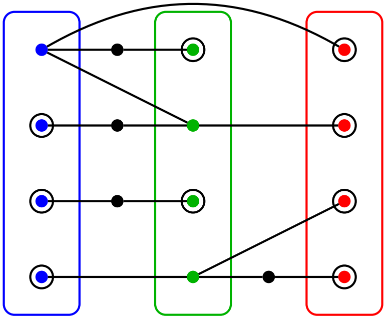

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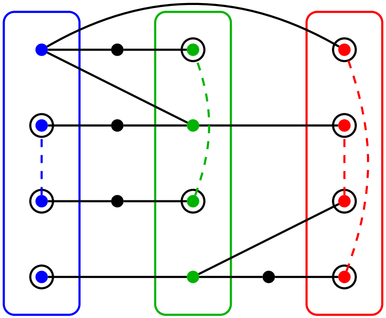

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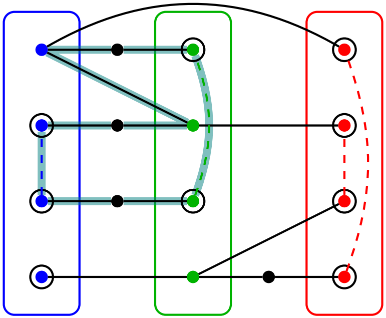

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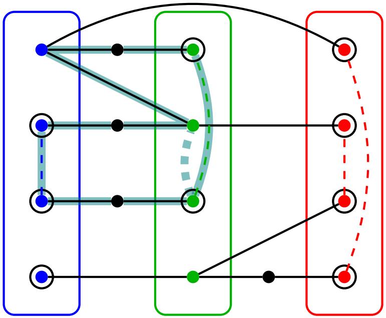

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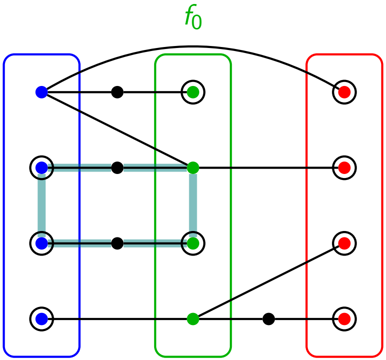

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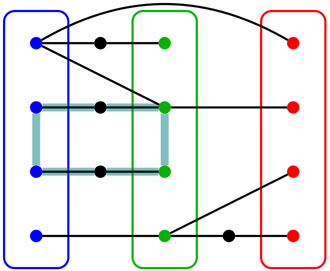
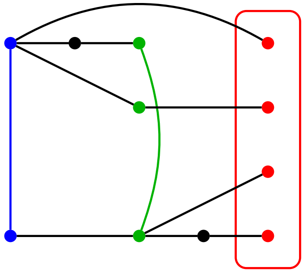
Removing the nice cycle





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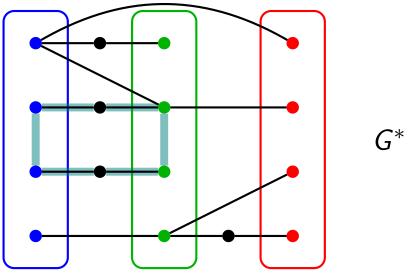

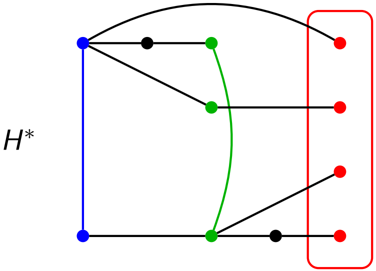

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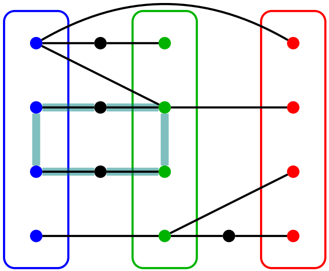
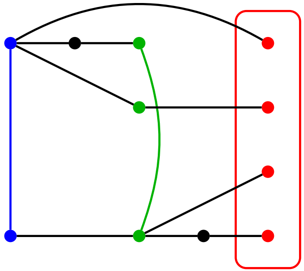
Removing the nice cycle



G^*



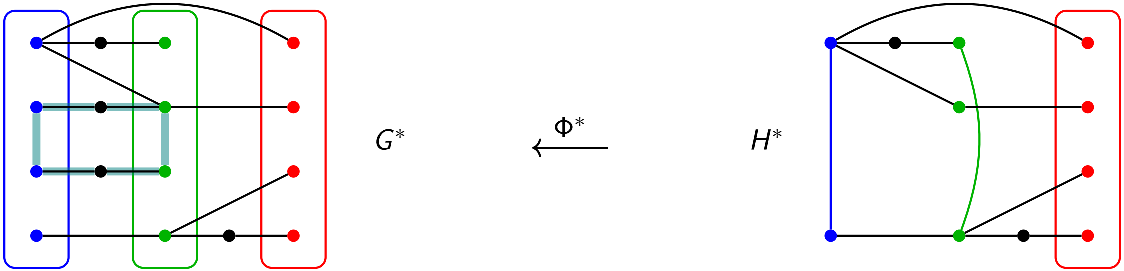
H^*

H^* is not a partial subhypergraph of G^*, but a **quasi-subhypergraph** of G^*.

Quasi-subhypergraphs arise from partial subhypergraphs by the following operations:

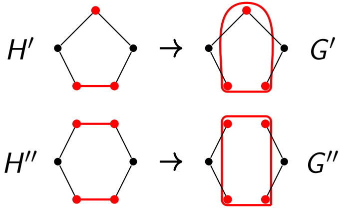
- **deleting** vertices from hyperedges
- **splitting** a hyperedge into a hypermatching

Removing the nice cycle



(H, Φ) is a **quasi-subhypergraph** of G ($(H, \Phi) \subseteq_Q G$) if:

- H is a hypergraph with $V(H) \subseteq V(G)$
- $\Phi\colon E_{\geq 1}(H) \to E(G)$
- $f \subseteq \Phi(f)$ for all $f \in E_{\geq 1}(H)$
- $\Phi^{-1}(e)$ is a hypermatching for all $e \in E(G)$



Showing that H^* has no odd cycle and no odd tree house

Goal: Show that H^* neither contains an odd cycle nor an odd tree house

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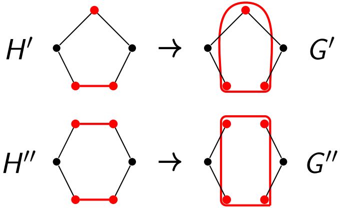
Odd cycles or odd tree houses in H^* are **quasi-subhypergraphs** of G^*.

→ To turn them into **partial subhypergraphs** of G^*, we have to handle **conflicts**.

Showing that H^* has no odd cycle and no odd tree house

conflict with respect to $(H, \Phi) \subseteq_Q G$:

"hyperedge in G that has been split or from which vertices have been removed to obtain H"



Showing that H^* has no odd cycle and no odd tree house

conflict with respect to $(H, \Phi) \subseteq_Q G$**:**
$e \in E(G)$ s.t. there is $f \in E_{\geq 1}(H)$ with $\Phi(f) = e$ and $f \subsetneq e \cap V(H)$.

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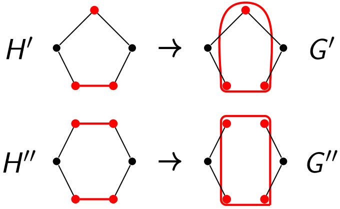

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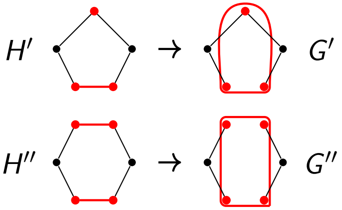
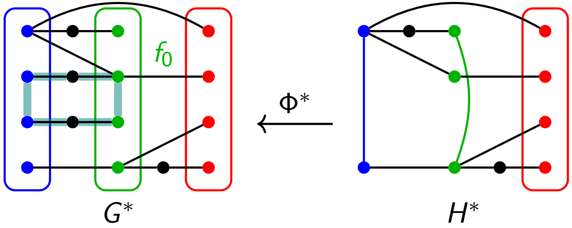


"hyperedge in G that has been split or from which vertices have been removed to obtain H"

nice cycle: even cycle $G^*[U, F] \subseteq_P G^*$ s.t.: For each hyperedge $e \in E_{\geq 4}(G^*) \cap F \setminus \{f_0\}$ and each $v \in e \cap U$, $\delta_{G^*}(v) \subseteq F$.



$\rightarrow$ f_0 is the only conflict w.r.t. $(H^*, \Phi^*) \subseteq_Q G^*$

H^* does not contain an odd tree house

Assume $T^* \subseteq_P H^*$ is an odd tree house.

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There is at most one conflict w.r.t. $(T^*, \Phi^* \restriction_{T^*})$ (namely, f_0).

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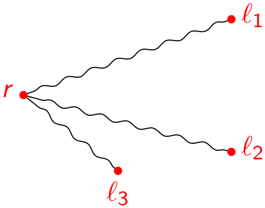
Strategy: Iteratively shortcut T^* until there is no more conflict.

H^* does not contain an odd tree house

Let $(T, \Phi) \subseteq_Q G^*$ s.t. T is an **odd tree house** with paths $(P_i)_{i\in[3]}$ and hyperedge $h = \{r, \ell_1, \ell_2, \ell_3\}$, there is **at most one conflict** and $|E(T)|$ is **minimum**.

conflict:
$e \in E(G)$ s.t. there is $f \in E(T)$ with $\Phi(f) = e$ and $f \subsetneq e \cap V(T)$.



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no conflict→ done

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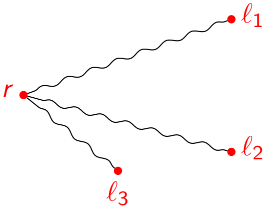

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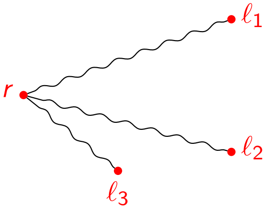

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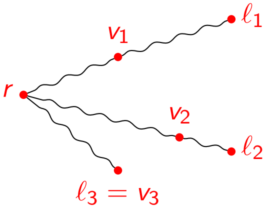

Case 1: $e = \Phi(h)$

$v_i :=$ next vertex from e on P_i after r

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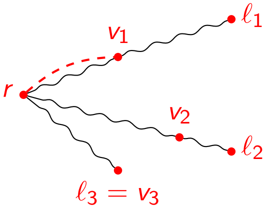

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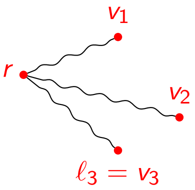

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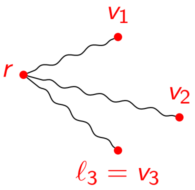

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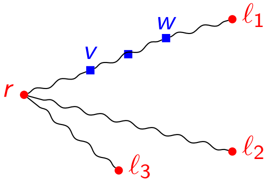

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w.l.o.g. $|e \cap V(P_1)| \geq 3$, let v and w be closest to r and ℓ_1

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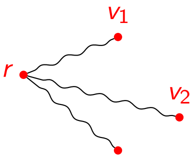

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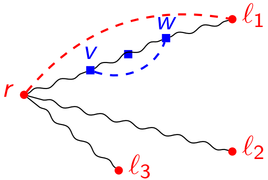

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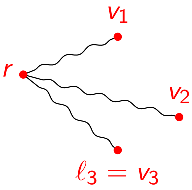

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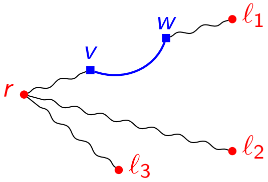

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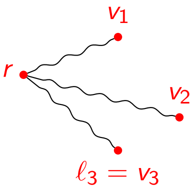

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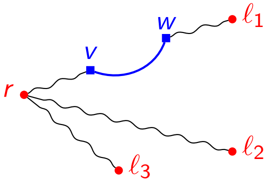

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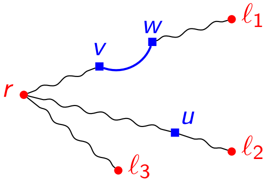

Case 3: $\forall i\colon |e \cap V(P_i)| \leq 2$

w.l.o.g. $\{v, w\} \in E(P_1)$ with $\Phi(\{v, w\}) = e$; $u \in e \cap V(P_2)$ closest to r

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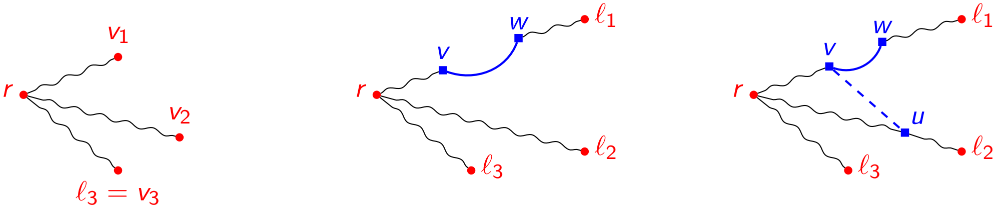

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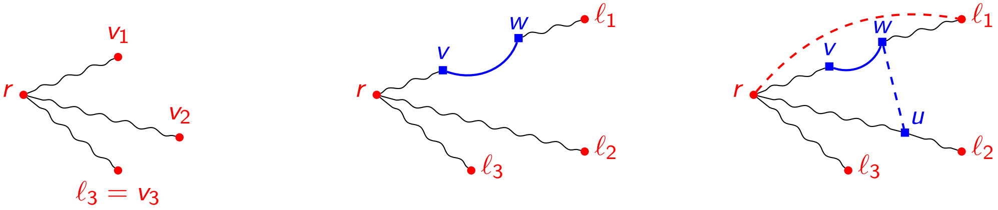

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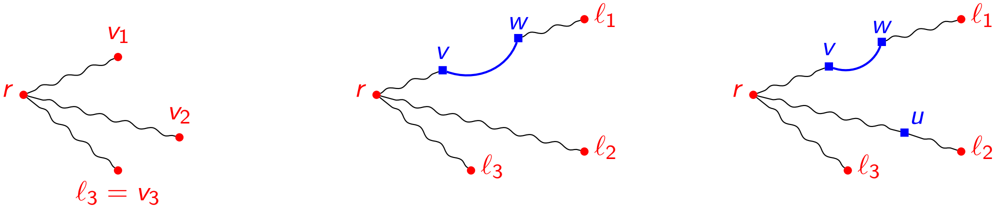

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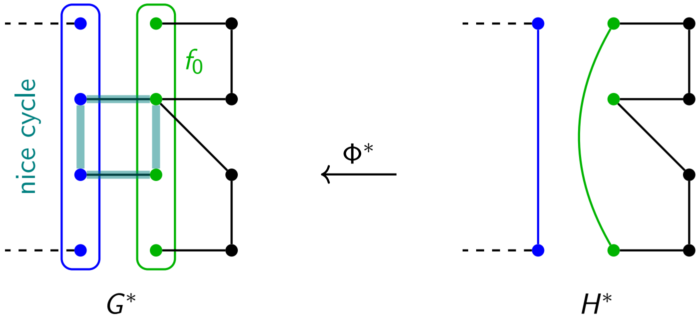
$\rightarrow P_1$ is even ↯

H^* does not contain an odd cycle



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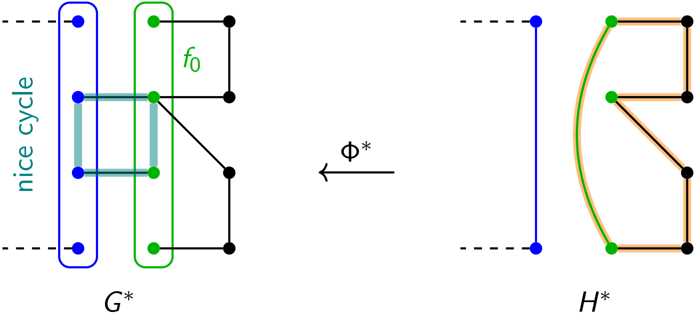

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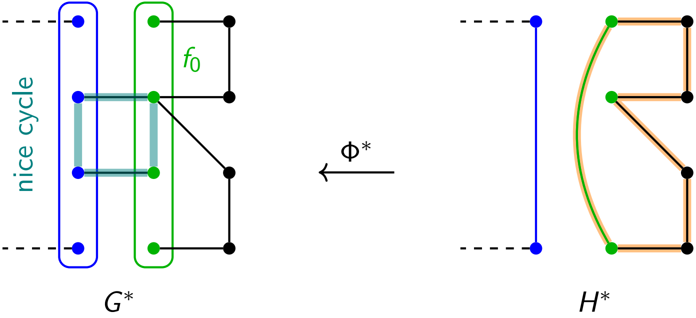

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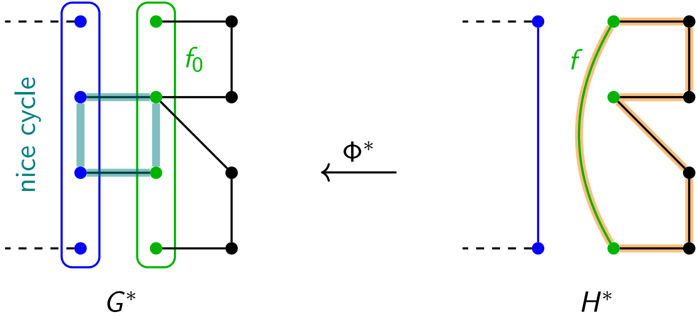

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consecutive vertices from f_0 have an odd distance on K^*

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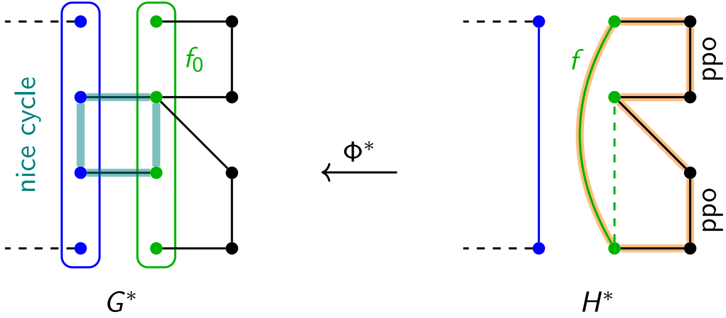

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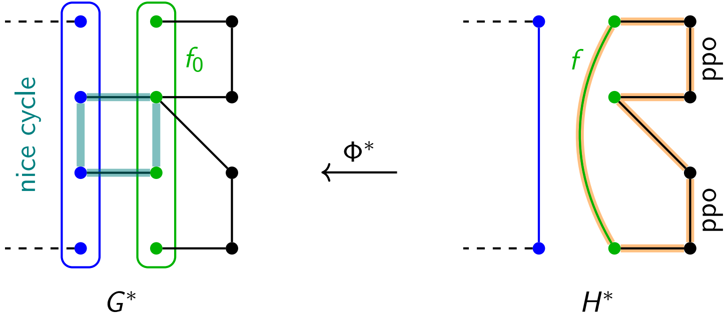

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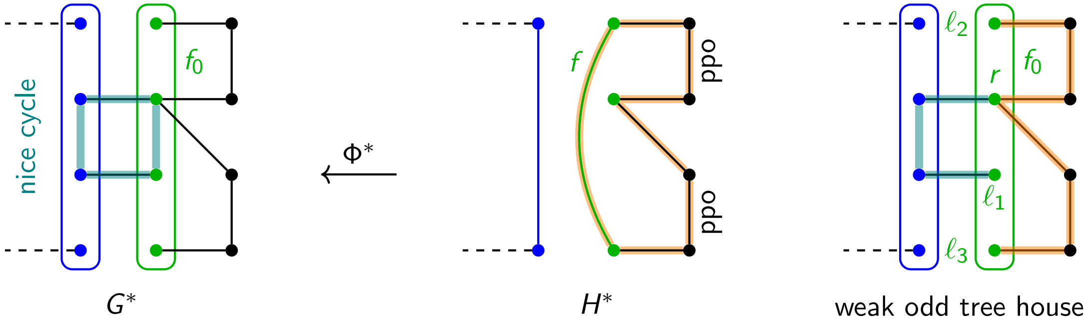
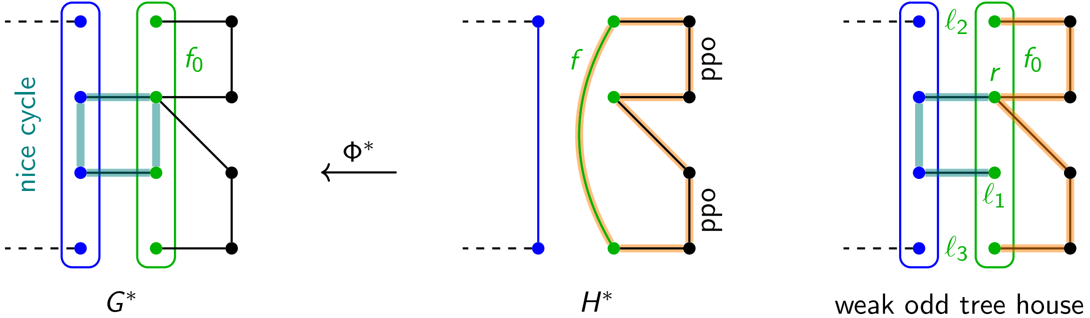


odd cycle + nice cycle $\rightsquigarrow$ **weak odd tree house** (paths can share inner vertices) $\subseteq_Q G^*$

H^* does not contain an odd cycle

Let $K^* = H^*[U, F] \subseteq_P H^*$ be an odd cycle.
$\to$ conflict w.r.t. $(K^*, \Phi^* \upharpoonright_{K^*})$
$\to \exists\ f \in F$ with $\Phi^*(f) = f_0$ and $3 = |f_0 \cap U|$
consecutive vertices from f_0 have an odd distance on K^*

conflict:
$e \in E(G)$ s.t. there is $f \in E(T)$ with $\Phi(f) = e$ and $f \subsetneq e \cap V(T)$.

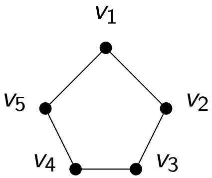


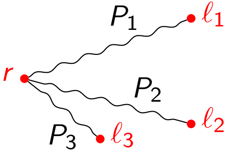
odd cycle + nice cycle $\rightsquigarrow$ **weak odd tree house** (paths can share inner vertices) $\subseteq_Q G^*$
analyze conflicts $\rightsquigarrow$ **odd cycle** or **odd tree house** $\subseteq_P G^*$

Unimodularity for disjoint hypergraphs

Theorem (Caoduro, N., Paat 25+)

Let G be a disjoint hypergraph. Then G is unimodular if and only if there does not exist $H \subseteq_P G$ that is an odd cycle or an odd tree house.





disjoint hypergraph: hyperedges of size ≥ 4 are pairwise disjoint

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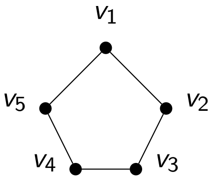

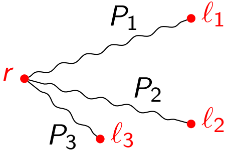

disjoint hypergraph: hyperedges of size ≥ 4 are pairwise disjoint

Next: extension to directed setting

Extension to directed disjoint hypergraphs

The family of incidence matrices of **directed hypergraphs** corresponds to $\{0, \pm 1\}$-matrices.

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Theorem (Caoduro, N., Paat '25+)

Let D be a disjoint directed hypergraph. Then $\boldsymbol{M}(D)$ is TU if and only if there does not exist $H \subseteq_P D$ that is a directed odd cycle or a directed odd tree house.

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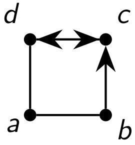
directed odd cycle: directed hypergraph C s.t.:

- underlying undirected hypergraph is a cycle
- $\Delta(\boldsymbol{M}(C)) = 2$



$$\begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Incidence matrices of directed odd cycles are called **unbalanced hole matrices**.

Extension to directed disjoint hypergraphs

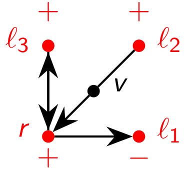
The family of incidence matrices of **directed hypergraphs** corresponds to $\{0, \pm 1\}$-matrices.

Theorem (Caoduro, N., Paat '25+)

Let D be a disjoint directed hypergraph. Then $\boldsymbol{M}(D)$ is TU if and only if there does not exist $H \subseteq_P D$ that is a directed odd cycle or a directed odd tree house.

directed odd tree house: directed hypergraph T s.t.:

- underlying undirected hypergraph is a "tree house" (r-ℓ_i-path P_i for $i \in [3]$ and $h = \{r, \ell_1, \ell_2, \ell_3\}$)
- $\Delta(\boldsymbol{M}(T)) = 2$



$$\begin{array}{c} r \\ \ell_1 \\ v \\ \ell_2 \\ \ell_3 \end{array} \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Application of our result

Conjecture (Padberg '88; Cornuéjols and Zuluaga '00)

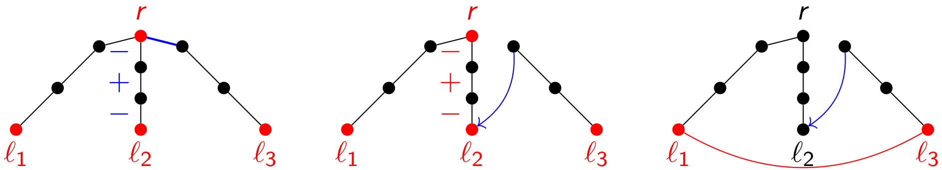
Given an almost TU matrix $\boldsymbol{A}$, there is a TU matrix $\boldsymbol{R}$ s.t. $\boldsymbol{AR}$ is an unbalanced hole matrix.

almost TU: minimally non-TU **unbalanced hole matrix:** $\boldsymbol{M}(C)$ for a directed odd cycle C

Corollary (Caoduro, N., Paat '25+)

The conjecture is true if $\boldsymbol{A}$ or $\boldsymbol{A}^T$ is the incidence matrix of a disjoint directed hypergraph.

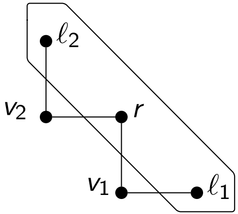
It suffices to check the conjecture for (directed odd cycles and) directed odd tree houses.



Future questions – beyond unimodularity

When is the incidence matrix of a disjoint hypergraph bimodular?

unlike the graph case: $\Delta(\boldsymbol{M(G)}) \geq 3 \not\Rightarrow$ two disjoint non-unimodular partial subhypergraphs



	$\{r, v_1\}$	$\{v_1, \ell_1\}$	$\{r, v_2\}$	$\{v_2, \ell_2\}$	e
r	1	0	1	0	1
v_1	1	1	0	0	0
ℓ_1	0	1	0	0	1
v_2	0	0	1	1	0
ℓ_2	0	0	0	1	1

Thank you!

↑

link to our ArXiv paper