An Exact Method for Nonlinear Network Flow Interdiction Problems

Martin Schmidt, Johannes Thürauf

Mixed Integer Programming European Workshop 2025
July 2, 2025

Joint work with Martin Schmidt

Home → SIAM Journal on Optimization → Vol. 34, Iss. 4 (2024) → 10.1137/22M152983X

Previous Article | Next Article

An Exact Method for Nonlinear Network Flow Interdiction Problems

Authors: Martin Schmidt and Johannes Thürauf | AUTHORS INFO & AFFILIATIONS

https://doi.org/10.1137/22M152983X

GET ACCESS | BibTeX | Tools

Abstract.

We study network flow interdiction problems with nonlinear and nonconvex flow models. The resulting model is a max-min bilevel optimization problem in which the follower's problem is nonlinear and nonconvex. In this game, the leader attacks a limited number of arcs with the goal of maximizing the load shed, and the follower aims at minimizing the load shed by solving a transport problem in the interdicted network. We develop an exact algorithm consisting of lower and upper bounding schemes that computes an optimal interdiction under the assumption that the interdicted network remains weakly connected. The main challenge consists of computing valid upper bounds for the maximal load shed, whereas lower bounds can directly be derived from the follower's problem. To compute an upper bound, we propose solving a specific bilevel problem, which is derived from restricting the flexibility of the follower when adjusting the load flow. This bilevel problem still has a nonlinear and nonconvex follower's problem, for which we then prove necessary and sufficient optimality conditions. Consequently, we obtain equivalent single-level reformulations of the specific bilevel model to compute upper bounds. Our numerical results show the applicability of this exact approach using the example of gas networks.

Overview

1. Introduction to Bilevel Optimization
2. Nonlinear Network Interdiction Problems
3. Computational Results

Introduction to Bilevel Optimization

General Bilevel Problems

$$
\begin{aligned}
\min_{x,y} \quad & F(x, y) \\
\text{s.t.} \quad & G(x, y) \geq 0, \\
& x \in X, \\
& y \in S(x),
\end{aligned}
\tag{UL}
$$

where $S(x)$ is the set of optimal solutions of the x-parameterized problem

$$
\begin{aligned}
\min_{y} \quad & f(x, y) \\
\text{s.t.} \quad & g(x, y) \geq 0, \\
& y \in Y.
\end{aligned}
\tag{LL}
$$

How to Solve a Bilevel Problem?

Reformulate the bilevel problem as a single-level problem

How to Solve a Bilevel Problem?

Reformulate the bilevel problem as a single-level problem

Common approaches

- Exploit optimality conditions for the lower-level problem (e.g., KKT conditions)
- Exploit strong-duality theorems (if at hand)
- Exploit the optimal value function of the lower-level problem

How to Solve a Bilevel Problem?

Reformulate the bilevel problem as a single-level problem

Common approaches

- Exploit optimality conditions for the lower-level problem (e.g., KKT conditions)
- Exploit strong-duality theorems (if at hand)
- Exploit the optimal value function of the lower-level problem

Which approach to choose?

$\rightarrow$ Depends on the problem at hand!

Linear Bilevel Problems

$$
\begin{aligned}
\min_{x,y} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \\
& y \in \arg\min_{\bar{y}} \left\{ f^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\}
\end{aligned}
$$

- Linear upper- and lower-level problem
- Duality theory available
- KKT conditions are necessary and sufficient

Linear Bilevel Problems: KKT Approach

Lower-level problem:

$$\begin{aligned} \min_{y} \quad & f^\top y \\ \text{s.t.} \quad & Dy \geq b - Cx. \end{aligned}$$

Lagrangian function:
$\mathcal{L}(x, y, \lambda) = f^\top y - \lambda^\top (Cx + Dy - b)$

Linear Bilevel Problems: KKT Approach

Lower-level problem:

$$\begin{aligned} \min_{y} \quad & f^\top y \\ \text{s.t.} \quad & Dy \geq b - Cx. \end{aligned}$$

Lagrangian function:
$\mathcal{L}(x, y, \lambda) = f^\top y - \lambda^\top (Cx + Dy - b)$

KKT conditions:

$$\begin{aligned} \nabla_y \mathcal{L}(x, y, \lambda) = f - D^\top \lambda = 0, \\ Cx + Dy \geq b, \\ \lambda \geq 0, \\ \lambda^\top (Cx + Dy - b) = 0. \end{aligned}$$

Linear Bilevel Problems: KKT Approach

$$\begin{aligned} \min_{x,y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & y \in \arg\min_{\bar{y}} \left\{ f^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\} \end{aligned}$$

$$\begin{aligned} \min_{x,y,\lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & Cx + Dy \geq b, \\ & D^\top \lambda = f, \ \lambda \geq 0, \\ & \lambda^\top (Cx + Dy - b) = 0. \end{aligned}$$

Linear Bilevel Problems: KKT Approach

$$\begin{aligned} \min_{x,y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & y \in \arg\min_{\bar{y}} \left\{ f^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\} \end{aligned}$$

$$\begin{aligned} \min_{x,y,\lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & Cx + Dy \geq b, \\ & D^\top \lambda = f, \ \lambda \geq 0, \\ & \lambda^\top (Cx + Dy - b) = 0. \end{aligned}$$

- Nonlinear and nonconvex complementarity constraints
- Apply Branch-and-Bound or Big-M reformulation

A More General Approach?

$$
\begin{aligned}
\min_{x,y} \quad & F(x,y) \\
\text{s.t.} \quad & G(x,y) \geq 0, \\
& x \in X, \\
& y \in S(x),
\end{aligned}
\tag{UL}
$$

where $S(x)$ is the set of optimal solutions of the x-parameterized problem

$$
\begin{aligned}
\min_{y} \quad & f(x,y) \\
\text{s.t.} \quad & g(x,y) \geq 0, \\
& y \in Y.
\end{aligned}
\tag{LL}
$$

Value-Function Reformulation

Lower-level optimal value function:

$$\varphi(x) := \min_y \{f(x,y) \colon g(x,y) \geq 0,\ y \in Y\}$$

Value-Function Reformulation

Lower-level optimal value function:

$$\varphi(x) := \min_y \{f(x,y) : g(x,y) \geq 0,\ y \in Y\}$$

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0, \\ & x \in X, \\ & y \in \arg\min_{\bar{y}} \{f(x,\bar{y}) : g(x,\bar{y}) \geq 0,\ \bar{y} \in Y\} \end{aligned}$$

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0, \\ & g(x,y) \geq 0, \\ & f(x,y) \leq \varphi(x), \\ & x \in X,\ y \in Y. \end{aligned}$$

Value-Function Reformulation

Benefit

- Applicable to many different problems without further assumptions

Value-Function Reformulation

Benefit

- Applicable to many different problems without further assumptions

Drawbacks

- Evaluating the value function is expensive
- Value function is generally not known in closed form
- Value function is generally nonsmooth (even under strong assumptions)

Value-Function Reformulation

Benefit

- Applicable to many different problems without further assumptions

Drawbacks

- Evaluating the value function is expensive
- Value function is generally not known in closed form
- Value function is generally nonsmooth (even under strong assumptions)

In practice often problem-specific knowledge is exploited to derive an explicit description of the optimal value function

Summary of General Solution Approaches

Linear and convex lower-level problems (under specific constraint qualifications)

- Strong duality approach
- KKT approach

→ Well-known and proven in practice since often straightforward to apply

Summary of General Solution Approaches

Linear and convex lower-level problems (under specific constraint qualifications)

- Strong duality approach
- KKT approach

→ Well-known and proven in practice since often straightforward to apply

Nonlinear and nonconvex lower-level problems

- Optimal value-function reformulation

→ Less proven in practice since generally difficult to apply

Leo Tolstoi: Anna Karenina

"Happy families are all alike; every unhappy family is unhappy in its own way."

Leo Tolstoi: Anna Karenina

"Happy families are all alike; every unhappy family is unhappy in its own way."

$$\text{happy} = \text{linear or convex}$$
$$\text{unhappy} = \text{nonlinear}$$

Nonlinear Network Interdiction Problems

Nonlinear Network Interdiction Problems

- Network $G = (V, A)$ with nodes V and arcs A
- Load flow $\ell \in \mathbb{R}^V$ representing injections and withdrawals at the nodes

Nonlinear Network Interdiction Problems

- Network $G = (V, A)$ with nodes V and arcs A
- Load flow $\ell \in \mathbb{R}^V$ representing injections and withdrawals at the nodes

Leader

- Destroys up to K many arcs of the network to maximize load shed, i.e., the amount of flow that cannot be served in the interdicted network

Nonlinear Network Interdiction Problems

- Network $G = (V, A)$ with nodes V and arcs A
- Load flow $\ell \in \mathbb{R}^V$ representing injections and withdrawals at the nodes

Leader

- Destroys up to K many arcs of the network to maximize load shed, i.e., the amount of flow that cannot be served in the interdicted network

Follower

- Minimize load shed and route the flow in the interdicted network

Nonlinear Network Interdiction Problems

- Network $G = (V, A)$ with nodes V and arcs A
- Load flow $\ell \in \mathbb{R}^V$ representing injections and withdrawals at the nodes

Leader

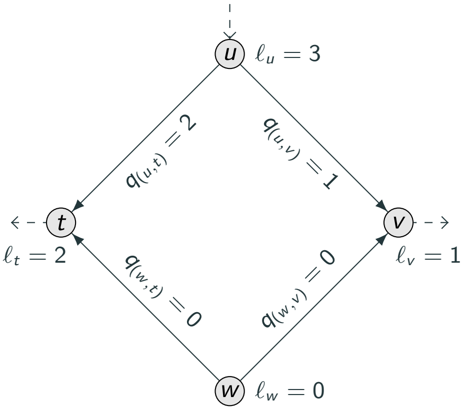
- Destroys up to K many arcs of the network to maximize load shed, i.e., the amount of flow that cannot be served in the interdicted network

Follower

- Minimize load shed and route the flow in the interdicted network

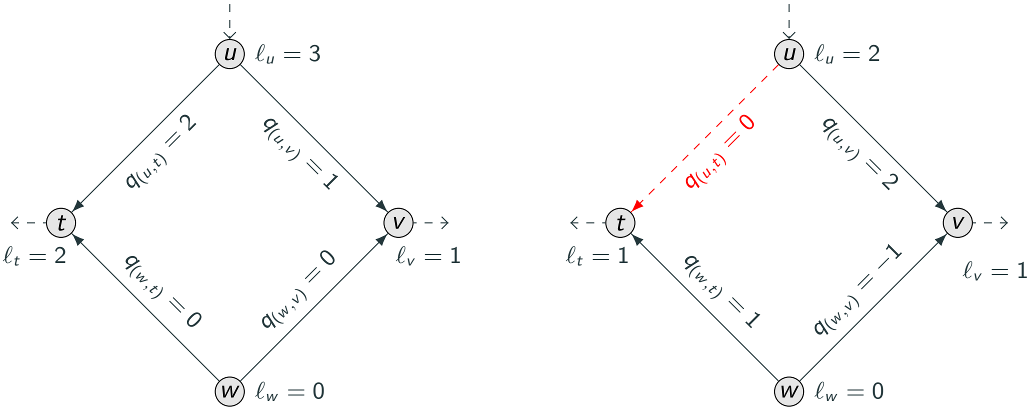
Max-min bilevel problem

Example: Linear Capacitated Flow Model



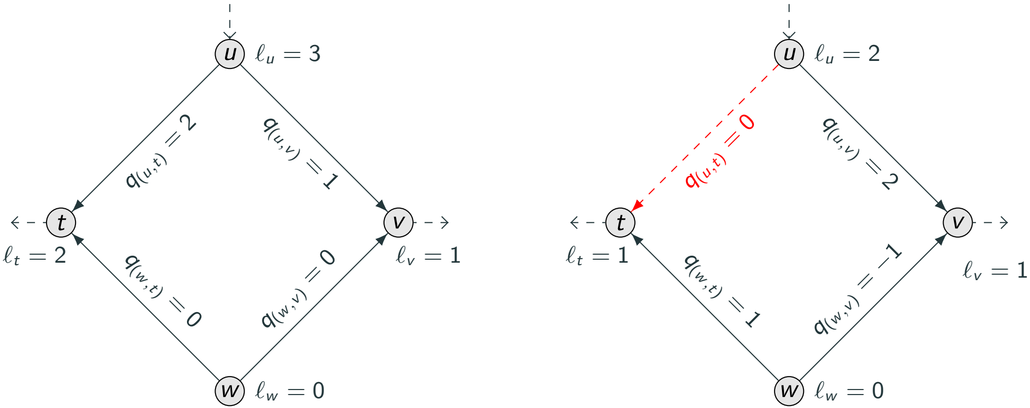
Capacitated linear flow model with $q_a^- = -2 < 2 = q^+$ and $K = 1$

Example: Linear Capacitated Flow Model



Capacitated linear flow model with $q_a^- = -2 < 2 = q^+$ and $K = 1$

Example: Linear Capacitated Flow Model



Capacitated linear flow model with $q_a^- = -2 < 2 = q^+$ and $K = 1$

Load shed of 1

Potential-Based Flows

Network modeled as a digraph $G = (V, A)$ with $V := V_+ \cup V_- \cup V_0$

Balanced load flow $\ell \in \mathbb{R}^V$, i.e., $\sum_{u \in V_+} \ell_u = \sum_{u \in V_-} \ell_u$, is feasible if $\exists\, q, \pi$ with

Potential-Based Flows

Network modeled as a digraph $G = (V, A)$ with $V := V_+ \cup V_- \cup V_0$

Balanced load flow $\ell \in \mathbb{R}^V$, i.e., $\sum_{u \in V_+} \ell_u = \sum_{u \in V_-} \ell_u$, is feasible if $\exists\, q, \pi$ with

$$\sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} \ell_v, & \text{if } v \in V_+ \\ -\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else,} \end{cases} \quad v \in V$$

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A$$

$$\pi_u - \pi_v = \Lambda_a \varphi(q_a), \quad a = (u, v) \in A$$

$$\pi_u^- \leq \pi_u \leq \pi_u^+, \quad u \in V$$

Potential-Based Flows

Network modeled as a digraph $G = (V, A)$ with $V := V_+ \cup V_- \cup V_0$

Balanced load flow $\ell \in \mathbb{R}^V$, i.e., $\sum_{u \in V_+} \ell_u = \sum_{u \in V_-} \ell_u$, is feasible if $\exists\, q, \pi$ with

$$\sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} \ell_v, & \text{if } v \in V_+ \\ -\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else,} \end{cases} \quad v \in V$$

$$q_a^- \le q_a \le q_a^+, \quad a \in A$$

$$\pi_u - \pi_v = \Lambda_a \varphi(q_a), \quad a = (u, v) \in A$$

$$\pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V$$

We consider potential functions of the form $\varphi(q_a) = q_a |q_a|^r$ with $r \ge 0$

$\rightarrow$ allows to model gas, hydrogen, water, and lossless DC power flow networks

Bilevel Model

$$\max_{x \in \{0,1\}^A} \quad \sum_{u \in V_-} \lambda_u \ell_u \quad \text{s.t.} \quad x \in X,\ (\lambda, q, \pi) \in S(x) \tag{UL}$$

Bilevel Model

$$\max_{x \in \{0,1\}^A} \quad \sum_{u \in V_-} \lambda_u \ell_u \quad \text{s.t.} \quad x \in X, \ (\lambda, q, \pi) \in S(x) \tag{UL}$$

$S(x)$: set of optimal solutions to the x-parameterized problem

$$\begin{aligned}
\min_{\lambda,q,\pi} \quad & \sum_{u \in V_-} \lambda_u \ell_u \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v (1 - \lambda_v) \ell_v, \quad v \in V \\
& (1 - x_a) q_a^- \le q_a \le (1 - x_a) q_a^+, \quad a \in A \\
& x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u, v) \in A \\
& \pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V \\
& 0 \le \lambda_u \le 1, \quad u \in V_-, \quad \lambda_u^- \le \lambda_u \le 1, \quad u \in V_+
\end{aligned} \tag{LL}$$

Literature Overview

Linear potential functions, e.g., $\varphi(q) = q$

$\rightarrow$ linear lower-level problem; see Salmeron et al. (2009), Johnson and Dey (2022)

Literature Overview

Linear potential functions, e.g., $\varphi(q) = q$

$\rightarrow$ linear lower-level problem; see Salmeron et al. (2009), Johnson and Dey (2022)

Nonlinear potential functions are a relative new field in the multilevel setting

$\rightarrow$ no duality theory applicable; see Mareldi et al. (2021) – restrictive assumptions

see Pfetsch and Schmitt (2022) – resilient networks

Linear potential functions, e.g., $\varphi(q) = q$

$\rightarrow$ linear lower-level problem; see Salmeron et al. (2009), Johnson and Dey (2022)

Nonlinear potential functions are a relative new field in the multilevel setting

$\rightarrow$ no duality theory applicable; see Mareldi et al. (2021) – restrictive assumptions

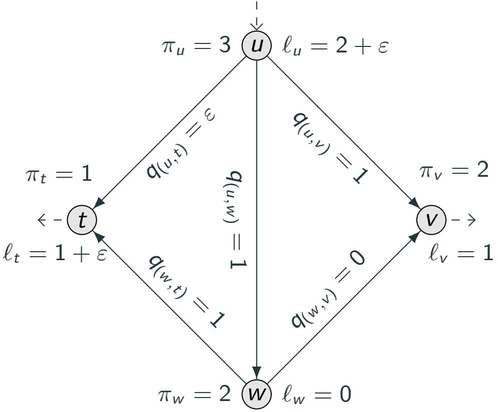
see Pfetsch and Schmitt (2022) – resilient networks

AC power flow networks

$\rightarrow$ not captured by nonlinear potential-based flows;

see, e.g., Bienstock and Abhinav (2010), Brian et al. (2021)

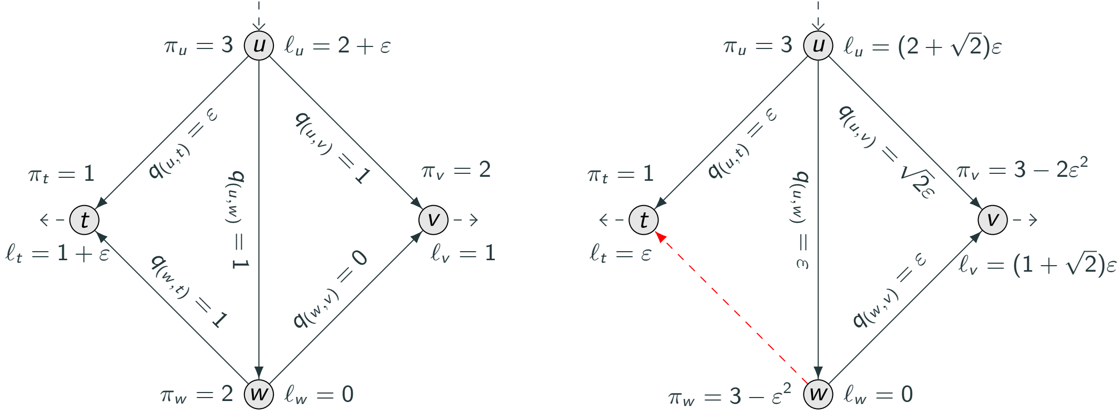
Why Potential-Based Flows Are Difficult



$$\varphi(q) = q|q|,\ \pi_i^+ = \infty,\ \pi_i^- = -\infty,\ i \in V, \quad \Lambda_a = 1, a \in A \setminus \{(u,t)\},\ \Lambda_{(u,t)} = 2/\varepsilon^2,$$

$$q_a^- = -1,\ q_a^+ = 1,\ a \in A \setminus \{(u,t),(w,v)\},\ q_{(u,t)}^- = q_{(w,v)}^- - \varepsilon,\ q_{(u,t)}^+ = q_{w,v}^+ = \varepsilon.$$

Why Potential-Based Flows Are Difficult



$$\varphi(q) = q|q|,\ \pi_i^+ = \infty,\ \pi_i^- = -\infty,\ i \in V, \quad \Lambda_a = 1, a \in A \setminus \{(u,t)\},\ \Lambda_{(u,t)} = 2/\varepsilon^2,$$

$$q_a^- = -1,\ q_a^+ = 1,\ a \in A \setminus \{(u,t),(w,v)\},\ q_{(u,t)}^- = q_{(w,v)}^- - \varepsilon,\ q_{(u,t)}^+ = q_{w,v}^+ = \varepsilon.$$

Load shed of $2 - \varepsilon(1 + \sqrt{2}) > 1$

General Algorithmic Idea

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, and $x^* \leftarrow 0$.

General Algorithmic Idea

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, and $x^* \leftarrow 0$.

While termination criterion is not satisfied

- Compute an upper bound $\phi^{\text{L}}_{\text{UB}}$ for the load shed and obtain the interdiction decision $x \in X$.
  if $\phi_{\text{UB}} > \phi^{\text{L}}_{\text{UB}}$ update upper bound

General Algorithmic Idea

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, and $x^* \leftarrow 0$.

While termination criterion is not satisfied

- Compute an upper bound $\phi_{\text{UB}}^{\text{L}}$ for the load shed and obtain the interdiction decision $x \in X$.
  if $\phi_{\text{UB}} > \phi_{\text{UB}}^{\text{L}}$ update upper bound

- Compute a lower bound $\phi_{\text{LB}}^{\text{F}}$ w.r.t. x.
  if $\phi_{\text{LB}}^{\text{F}} > \phi_{\text{LB}}$ update lower bound and best known interdiction x^*

General Algorithmic Idea

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, and $x^* \leftarrow 0$.

While termination criterion is not satisfied

- Compute an upper bound $\phi^{\text{L}}_{\text{UB}}$ for the load shed and obtain the interdiction decision $x \in X$.
  if $\phi_{\text{UB}} > \phi^{\text{L}}_{\text{UB}}$ update upper bound

- Compute a lower bound $\phi^{\text{F}}_{\text{LB}}$ w.r.t. x.
  if $\phi^{\text{F}}_{\text{LB}} > \phi_{\text{LB}}$ update lower bound and best known interdiction x^*

- Add the no-good cut to cut off the current interdiction x

$$\sum_{a \in A : x_a = 1} (1 - x) + \sum_{a \in A : x_a = 0} x \geq 1$$

General Algorithmic Idea

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, and $x^* \leftarrow 0$.

While termination criterion is not satisfied

- Compute an upper bound $\phi^{\text{L}}_{\text{UB}}$ for the load shed and obtain the interdiction decision $x \in X$.
  if $\phi_{\text{UB}} > \phi^{\text{L}}_{\text{UB}}$ update upper bound
- Compute a lower bound $\phi^{\text{F}}_{\text{LB}}$ w.r.t. x.
  if $\phi^{\text{F}}_{\text{LB}} > \phi_{\text{LB}}$ update lower bound and best known interdiction x^*
- Add the no-good cut to cut off the current interdiction x

$$\sum_{a \in A : x_a = 1} (1 - x) + \sum_{a \in A : x_a = 0} x \geq 1$$

Computing a Lower Bound

Compute a lower bound for a fixed interdiction decision $x \in X$

Computing a Lower Bound

Compute a lower bound for a fixed interdiction decision $x \in X$

- Solve the follower's problem for fixed interdiction decision x (NLP)
- Obtain lower bound for load shed and "best response" $(1 - \lambda) \circ \ell$ of the follower

Computing a Lower Bound

Compute a lower bound for a fixed interdiction decision $x \in X$

- Solve the follower's problem for fixed interdiction decision x (NLP)
- Obtain lower bound for load shed and "best response" $(1 - \lambda) \circ \ell$ of the follower

The real challenge consists of computing a "good" upper bound

Computing an Upper Bound

Assumption: For fixed $x \in X$, the interdicted network $G(x)$ is weakly connected.

Computing an Upper Bound

Assumption: For fixed $x \in X$, the interdicted network $G(x)$ is weakly connected.

$$
\begin{aligned}
\max_{x \in X} \ \min_{\lambda, q, \pi} \quad & \sum_{u \in V_-} \lambda_u \ell_u \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v (1 - \lambda_v) \ell_v, \quad v \in V \\
& (1 - x_a) q_a^- \le q_a \le (1 - x_a) q_a^+, \quad a \in A \\
& x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u, v) \in A \\
& \pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V \\
& 0 \le \lambda_u \le 1, \quad u \in V_-, \quad \lambda_u^- \le \lambda_u \le 1, \quad u \in V_+
\end{aligned}
$$

Computing an Upper Bound

Assumption: For fixed $x \in X$, the interdicted network $G(x)$ is weakly connected.

$$
\begin{aligned}
\max_{x \in X} \ \min_{\lambda, q, \pi} \quad & \sum_{u \in V_-} \lambda_u \ell_u \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v (1 - \lambda_v) \ell_v, \quad v \in V \\
& (1 - x_a) q_a^- \le q_a \le (1 - x_a) q_a^+, \quad a \in A \\
& x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u, v) \in A \\
& \pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V \\
& 0 \le \lambda_u \le 1, \quad u \in V_-, \quad \lambda_u^- \le \lambda_u \le 1, \quad u \in V_+
\end{aligned}
$$

Key idea: $\lambda_u = \lambda_v,\ u, v \in V$, i.e., replace $\lambda \in \mathbb{R}^V$ by $\lambda \in \mathbb{R}$

Computing an Upper Bound

$$\max_{x \in X} \min_{\lambda, q, \pi} \sum_{u \in V_-} \lambda \ell_u$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v (1 - \lambda) \ell_v, \quad v \in V$$

$$(1 - x_a) q_a^- \leq q_a \leq (1 - x_a) q_a^+, \quad a \in A$$

$$x_a M_a^- \leq \pi_u - \pi_v - \Lambda_a \varphi(q_a) \leq x_a M_a^+, \quad a = (u, v) \in A$$

$$\pi_u^- \leq \pi_u \leq \pi_u^+, \quad u \in V,$$

$$0 \leq \lambda_u \leq 1, \ u \in V_-, \quad \lambda_u^- \leq \lambda_u \leq 1, \ u \in V_+$$

Restrict the follower $\rightarrow$ solving this bilevel problem yields an upper bound

Computing an Upper Bound

$$
\begin{aligned}
\max_{x \in X} \ \min_{\lambda, q, \pi} \quad & \sum_{u \in V_-} \lambda \ell_u \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v (1 - \lambda) \ell_v, \quad v \in V \\
& (1 - x_a) q_a^- \le q_a \le (1 - x_a) q_a^+, \quad a \in A \\
& x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u, v) \in A \\
& \pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V, \\
& 0 \le \lambda_u \le 1, \ u \in V_-, \quad \lambda_u^- \le \lambda_u \le 1, \ u \in V_+
\end{aligned}
$$

Restrict the follower $\rightarrow$ solving this bilevel problem yields an upper bound

Bilevel problem still has a nonlinear and nonconvex follower's problem

Properties of Potential-Based Flows

Let a balanced load flow $\ell \in \mathbb{R}^V$ be given and let's ignore potential and flow bounds.

Properties of Potential-Based Flows

Let a balanced load flow $\ell \in \mathbb{R}^V$ be given and let's ignore potential and flow bounds.

Uniqueness results: Maugis (1977) , Collins et al. (1978)

There exist feasible potentials $\pi \in \mathbb{R}^V$ and unique flows $q \in \mathbb{R}^A$ so that the set of feasible points is given by

$$\{(q, \tilde{\pi}) \,:\, \tilde{\pi} = \pi + \tau \mathbb{1},\ \tau \in \mathbb{R}\}\,.$$

Properties of Potential-Based Flows

Let a balanced load flow $\ell \in \mathbb{R}^V$ be given and let's ignore potential and flow bounds.

Uniqueness results: Maugis (1977) , Collins et al. (1978)

There exist feasible potentials $\pi \in \mathbb{R}^V$ and unique flows $q \in \mathbb{R}^A$ so that the set of feasible points is given by

$$\{(q, \tilde{\pi}) \, : \, \tilde{\pi} = \pi + \tau \mathbb{1}, \ \tau \in \mathbb{R}\} \, .$$

Positive Homogeneity: Gross et al. (2019)

If (q, π) is feasible w.r.t. load flow ℓ, then for any $\lambda \in \mathbb{R}$ the point $(\lambda q, \lambda^r \pi)$ is feasible w.r.t. $\lambda \ell$.

Bilevel Reformulation

$$\begin{aligned} \max_{x,q,\pi} \quad & \sum_{u \in V_-} \lambda \ell_u \\ \text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v \ell_v, \quad v \in V \\ & x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u,v) \in A \\ & -(1 - x_a) Q \le q_a \le (1 - x_a) Q, \quad a \in A \\ & x \in X, \ (\lambda, \tau) \in S(x, q, \pi) \end{aligned} \tag{UL}$$

$S(x, q, \pi)$ consists of all optimal solutions of

$$\begin{aligned} \min_{\lambda \in [0,1], \tau \in \mathbb{R}} \quad & \sum_{u \in V_-} \lambda \ell_u \\ \text{s.t.} \quad & \pi_u^- \le (1 - \lambda)^r \pi_u + \tau \le \pi_u^+, \quad u \in V \\ & (1 - x_a) q_a^- \le (1 - \lambda) q_a \le (1 - x_a) q_a^+, \quad a \in A \end{aligned} \tag{LL}$$

Necessary and Sufficient Optimality Condition

Theorem

For any feasible upper-level decision, the point (λ, τ) is an optimal solution of the follower's problem if and only if (λ, τ) is feasible for the follower's problem and the point $(x, q, \pi, \lambda, \tau)$ satisfies at least one of the following conditions.

(i) *There are nodes $u, v \in V$ such that the corresponding lower and upper potential bound are tight, i.e.,*

$$(1-\lambda)^r \pi_u + \tau = \pi_u^+, \quad (1-\lambda)^r \pi_v + \tau = \pi_v^- .$$

(ii) *There is an arc $a \in A$ such that the lower or upper flow bound is tight, i.e.,*

$$(1-\lambda) q_a = q_a^- \ \text{ or } \ (1-\lambda) q_a = q_a^+ .$$

(iii) *There is no load shed, i.e., $\lambda = 0$.*

Single-Level Reformulation

$$\max_{\substack{x,\lambda,q,\pi,\tilde{\varepsilon}^+,\tilde{\varepsilon}^-,\varepsilon^+,\\ \varepsilon^-,\bar{y},\underline{y},\tilde{y},y'}} \quad \sum_{u \in V_-} \lambda \ell_u$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v \ell_v, \quad v \in V$$

$$x_a M_a^- \leq \pi_u - \pi_v - \Lambda_a \varphi(q_a) \leq x_a M_a^+, \quad a = (u, v) \in A$$

$$-(1 - x_a) Q \leq q_a \leq (1 - x_a) Q, \quad a \in A$$

Necessary and sufficient optimality condition of the follower

$\rightarrow$ additional $2|A| + 2|V| + 1$ binary variables

Single-Level Reformulation

$$\max_{\substack{x,\lambda,q,\pi,\tilde{\varepsilon}^+,\tilde{\varepsilon}^-,\varepsilon^+,\\ \varepsilon^-,\bar{y},\underline{y},\tilde{y},y'}} \quad \sum_{u \in V_-} \lambda \ell_u$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v \ell_v, \quad v \in V$$

$$x_a M_a^- \leq \pi_u - \pi_v - \Lambda_a \varphi(q_a) \leq x_a M_a^+, \quad a = (u, v) \in A$$

$$-(1 - x_a)Q \leq q_a \leq (1 - x_a)Q, \quad a \in A$$

Necessary and sufficient optimality condition of the follower

$\rightarrow$ additional $2|A| + 2|V| + 1$ binary variables

Single-level reformulation: mixed-integer nonlinear optimization problem
→ can be “solved” by state-of-the-art optimization solvers, e.g., SCIP

Single-Level Reformulation

$$
\begin{aligned}
\max_{\substack{x,\lambda,q,\pi,\tilde{\varepsilon}^+,\tilde{\varepsilon}^-,\varepsilon^+,\\ \varepsilon^-,\bar{y},\underline{y},\tilde{y},y'}} \quad & \sum_{u \in V_-} \lambda \ell_u \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \sigma_v \ell_v, \quad v \in V \\
& x_a M_a^- \leq \pi_u - \pi_v - \Lambda_a \varphi(q_a) \leq x_a M_a^+, \quad a = (u,v) \in A \\
& -(1 - x_a) Q \leq q_a \leq (1 - x_a) Q, \quad a \in A \\
& \text{Necessary and sufficient optimality condition of the follower} \\
& \to \text{ additional } 2\,|A| + 2\,|V| + 1 \text{ binary variables}
\end{aligned}
$$

Single-level reformulation: mixed-integer nonlinear optimization problem
→ can be “solved” by state-of-the-art optimization solvers, e.g., SCIP

Further single-level reformulations (R1) and (R2) with less nonlinear terms

Exact Algorithm

Algorithm 1: Solving potential-based network flow interdiction problems

Input: load flow $\ell \in \mathbb{R}^V$ and optimality tolerance $\varepsilon \geq 0$.

Initialize $\phi_{\text{LB}} \leftarrow 0$, $\phi_{\text{UB}} \leftarrow \infty$, $\ell^* \leftarrow \ell$, and $x^* \leftarrow 0$.

while $(\phi_{\text{UB}} - \phi_{\text{LB}})/\phi_{\text{LB}} \geq \varepsilon$ **do**

- Solve single-level reformulation (R1) or (R2) w.r.t. ℓ^* and obtain the interdiction decision $x \in X$ and the objective value $\phi^{\text{L}}_{\text{UB}}$.
- **if** Problem (R1) or (R2) w.r.t. ℓ^* is infeasible **then**
  - Update $\phi_{\text{UB}} \leftarrow \phi_{\text{LB}}$ and **return** interdiction $x^* \in X$.
- **if** $\phi_{\text{UB}} > \phi^{\text{L}}_{\text{UB}} + \phi_{\text{LB}}$ **then** Update $\phi_{\text{UB}} \leftarrow \phi^{\text{L}}_{\text{UB}} + \phi_{\text{LB}}$.
- Solve the lower-level problem w.r.t. x and ℓ. Obtain solution (λ, q, π) with objective value $\phi^{\text{F}}_{\text{LB}}$.
- **if** $\phi^{\text{F}}_{\text{LB}} > \phi_{\text{LB}}$ **then**
  - Update $\phi_{\text{LB}} \leftarrow \phi^{\text{F}}_{\text{LB}}$, $\ell^* \leftarrow (1 - \lambda) \circ \ell$, and set $x^* = x$.
- Add the no-good cut to X to cut off the current interdiction x.

return interdiction $x^* \in X$

Computational Results

Computational Results

Computational setup

- Python 3.7 using Pyomo 6.4.2.dev0
- Solver SCIP 8.0.0 with Gurobi 9.0.3 and Ipopt 3.14.4
- Server with XEON_SP_6126 CPU and 16 GB RAM
- Timelimit of 24h

Instances

- GasLib 40 consisting of 40 nodes (3 sources and 29 sinks) and 39 pipes
- $\varphi(q) = q|q|$
- Different load flows for injections and withdrawals

Enumeration approach

- Solve a MIP to find a feasible interdiction decision $x \in X$
- Solve the follower's problem w.r.t. x
- Add no-good cut and find next interdiction decision until no feasible interdiction decision is left

Runtimes and Number of Iterations

Table 1: Runtimes and number of iterations for GasLib-40 and two different loadflows

	$K=1$		$K=2$		$K=3$		$K=4$		$K=5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	33.2	26	268.1	270	1760.1	1619	14 237.1	5893	—	—
(R1)	57.0	2	172.9	3	1167.8	11	18 202.0	152	—	—
(R2)	31.2	2	107.1	3	701.8	11	17 583.6	155	72 505.0	725

	$K=1$		$K=2$		$K=3$		$K=4$		$K=5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	16.4	26	195.2	270	1673.8	1619	13 883.7	5893	86 125.6	13 221
(R1)	87.6	3	86.7	2	3013.0	24	24 299.6	185	—	—
(R2)	46.6	3	108.9	2	1375.3	24	—	—	—	—

Adapted Algorithm

Solving MINLP to optimality is much harder than solving NLPs

Adapted Algorithm

Solving MINLP to optimality is much harder than solving NLPs

Adapted approach

- Reformulated single-level reformulation
  $\rightarrow$ each feasible solution leads to an interdiction decision with positive load shed

Adapted Algorithm

Solving MINLP to optimality is much harder than solving NLPs

Adapted approach

- Reformulated single-level reformulation
  $\rightarrow$ each feasible solution leads to an interdiction decision with positive load shed
- Solve only the first and each *i*th iteration to optimality
  $\rightarrow$ leads to an upper bound

Adapted Algorithm

Solving MINLP to optimality is much harder than solving NLPs

Adapted approach

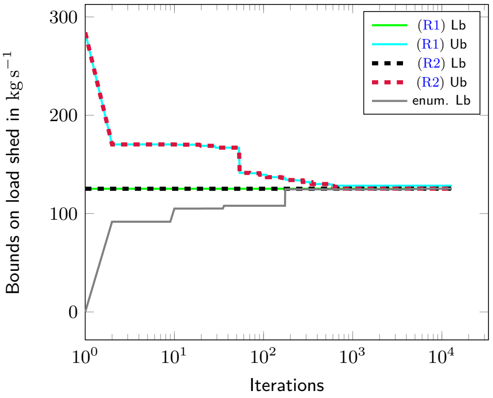
- Reformulated single-level reformulation
  $\rightarrow$ each feasible solution leads to an interdiction decision with positive load shed
- Solve only the first and each ith iteration to optimality
  $\rightarrow$ leads to an upper bound
- Compute only a feasible point in the remaining iterations
  $\rightarrow$ leads to interdiction decisions that violate current best response of the follower

Runtimes and Number of Iterations

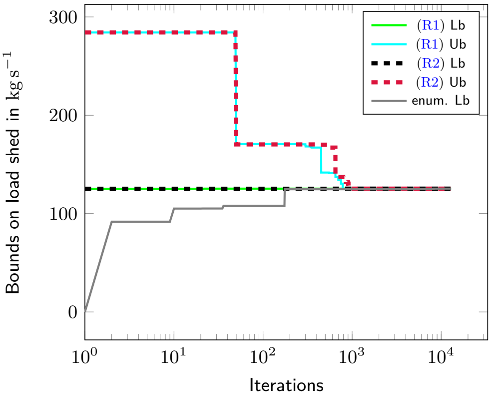
	$K = 1$		$K = 2$		$K = 3$		$K = 4$		$K = 5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	33.2	26	268.1	270	1760.1	1619	14 237.1	5893	—	—
(R1)	53.5	2	122.4	3	533.2	11	4938.1	162	36 468.7	784
(R2)	37.8	2	89.4	3	365.3	12	3976.1	161	21 256.1	912

	$K = 1$		$K = 2$		$K = 3$		$K = 4$		$K = 5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	16.4	26	195.2	270	1673.8	1619	13 883.7	5893	86 125.6	13 221
(R1)	76.3	3	96.3	2	946.5	24	7922.4	194	46 487.9	855
(R2)	42.6	3	79.3	2	760.7	24	5703.4	194	37 273.9	876

Convergence of Lower and Upper Bounds: GasLib40 for Budget $K = 5$



Single-level reformulations solved to optimality in each iteration.



First and each 50th iteration is solved to optimality

Conclusion

An exact method for nonlinear network interdiction problems

- Bilevel approach to compute an upper bound
- Necessary and sufficient optimality condition for a nonlinear and nonconvex lower-level problem
- Promising computational results

Conclusion

An exact method for nonlinear network interdiction problems

- Bilevel approach to compute an upper bound
- Necessary and sufficient optimality condition for a nonlinear and nonconvex lower-level problem
- Promising computational results

Future research

- Include active elements such as compressors
- Drop the assumption that the interdicted network is weakly connected
- Use convex relaxations for computing the upper bound

BOBILib: Bilevel Optimization (Benchmark) Instance Library

- More than 2600 instances of mixed-integer linear bilevel optimization problems
- Well-curated set of test instances
- Freely available for the research community
- Testing of new methods + comparison with other ones
- Different types of instances
    - Interdiction
    - Mixed-integer
    - Pure integer
- Benchmark sets for all of them
- Extensive numerical results
- New data + solution format
- All best known solutions available

`https://bobilib.org`