Adjustable Robust Mixed-Integer Nonlinear Network Design

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Mixed Integer Programming European Workshop 2025
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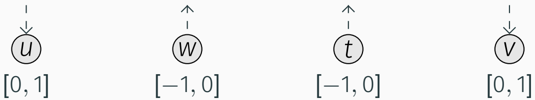
Resilient Energy Networks

Resilient Energy Networks

Two expansion options

- Large capacity arc (expensive)
- Small capacity arc (cheap)



Design cost-efficient resilient network

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Easy → maximal injections and withdrawals is the worst-case?

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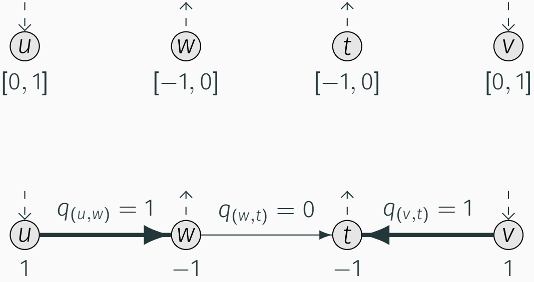

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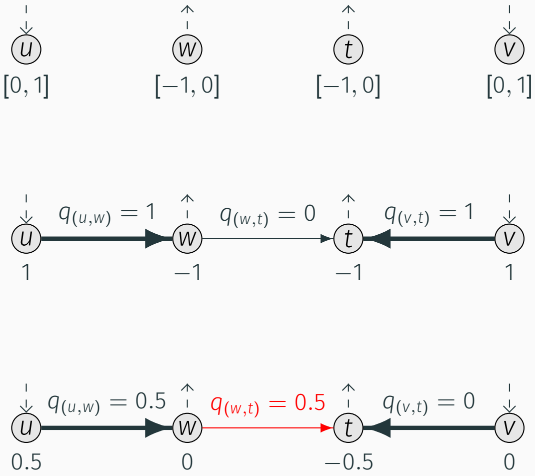
Wrong!



Task

Compute a network design taking into account demand uncertainties

Consider an accurate nonconvex transport model

$\rightarrow$ Adjustable robust MINLP

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Discrete decisions and nonlinear constraints

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Key Components of the Solution Approach

Exploit the underlying network and structural properties of potential-based flows

Overview

Potential-Based Flows

Robust Network Design Model

Characterizing Worst-Case Scenarios

Computational Results

Potential-Based Flows

Potential-Based Flows

Network modeled as a digraph $G = (V, A)$ with $V := V_+ \cup V_- \cup V_0$

Balanced load flow $\ell \in \mathbb{R}^V$, i.e., $\sum_{u \in V_+} \ell_u = \sum_{u \in V_-} \ell_u$, is feasible if $\exists\, q, \pi$ with

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$$\sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} \ell_v, & \text{if } v \in V_+ \\ -\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else} \end{cases} \quad v \in V$$

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A$$

$$\pi_u - \pi_v = \Lambda_a \varphi(q_a), \quad a = (u, v) \in A$$

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We consider potential functions of the form $\varphi(q_a) = q_a |q_a|^r$ with $r \geq 0$

$\rightarrow$ allows to model gas, hydrogen, water, and lossless DC power flow networks

Properties of Potential-Based Flows

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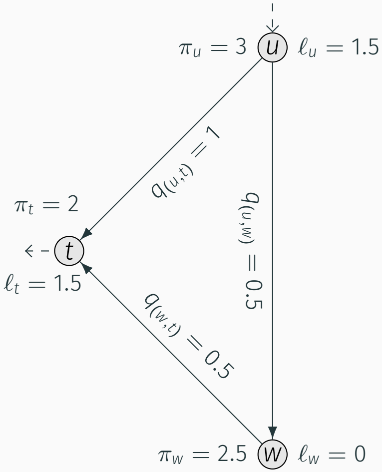
Uniqueness results: Maugis (1977) , Collins et al. (1978)

There exist feasible potentials $\pi \in \mathbb{R}^V$ and unique flows $q \in \mathbb{R}^A$ so that the set of feasible points is given by

$$\{(q, \tilde{\pi}) \, : \, \tilde{\pi} = \pi + \tau \mathbb{1}, \; \tau \in \mathbb{R}\} \, .$$



$$\varphi(q) = q, \quad \Lambda_a = 1,\ a \in A$$

$$q_a^- = -3,\ q_a^+ = 3,\ a \in A$$

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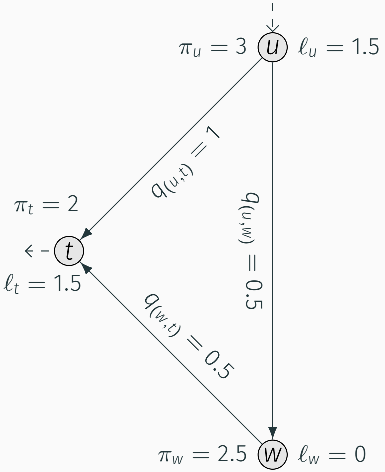
Why Potential-Based Flows Are Difficult



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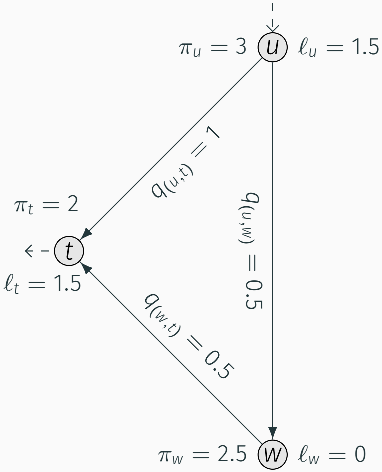

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$$\pi_u - \pi_t = \pi_u - \pi_w + \pi_w - \pi_t =$$
$$= q_{u,t} = q_{u,w} + q_{t,w}$$

Robust Network Design Model

Network Expansion

Expansion variables $x_a \in \{0, 1\}$ for $a \in A_{\text{ca}}$

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Mixed-integer nonconvex optimization problem

$$
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\min_{x,q,\pi} \quad & \sum_{a \in A_{\mathrm{ca}}} c_a x_a \\
\text{s.t.} \quad & x \in X \subseteq \{0,1\}^{A_{\mathrm{ca}}} \\
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& \text{potential-based flows}(q,\pi), \quad a \in A \\
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Nominal Network Design: Model

Mixed-integer nonconvex optimization problem

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Demand fluctuations can lead to infeasibility of the computed network design!

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Demand fluctuations can lead to infeasibility of the computed network design!
$\rightarrow$ consider demand uncertainties

Modeling Demand Uncertainty

Robust optimization approach
→ Protect against all demand fluctuations within the uncertainty set

$$U := \left\{ \ell \in \mathbb{R}_{\geq 0} : \sum_{u \in V_+} \ell_u = \sum_{u \in V_-} \ell_u,\ \ell_u = 0\ \forall u \in V_0 \right\} \cap Z$$

with Z being a compact set

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General form of the uncertainty set: polyhedral, ellipsoidal, ...
$\rightarrow$ covers different situations of demand uncertainties

Robust Network Design

Adjustable robust nonconvex optimization problem:

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How can we solve this challenging problem?

Literature Overview: Robust Network Design

Static routing

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Adjustable robust network design with nonlinear flows

- Gas networks: Sundar et al. 2021 only uncertain sinks and unlimited sources
- Tree-shaped networks: Robinius et al. 2019
- Arc failures Pfetsch and Schmitt 2023

Adversarial Solution Approach

General Algorithmic Idea

Determine a set of finitely many scenarios $S \subseteq U$

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Well known approach in strictly robust optimization; see e.g., Yanıkoğlu et al. 2019

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How can we find violating scenarios for the adjustable robust nonconvex problem?

Can we guarantee finite termination?

Characterizing Worst-Case Scenarios

Finding Worst-Case Scenarios

Three types of “worst-case” scenarios

- Unbalanced demands between different connected components
- Violating flow bounds
- Violating potential bounds

Worst-Case Scenarios: Unbalanced Demands

Fixed network expansion $x \in X$ and the expanded graph $G(x) = (V, A(x))$

Connected component $G^i = (V^i, A^i)$

Worst-Case Scenarios: Unbalanced Demands

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Connected component $G^i = (V^i, A^i)$

Find unbalanced demands

$$\mu_{G^i}(x) := \max_{\ell} |y| \quad \text{s.t.} \quad y = \sum_{u \in V^i \cap V_+} \ell_u - \sum_{u \in V^i \cap V_-} \ell_u, \; \ell \in U$$

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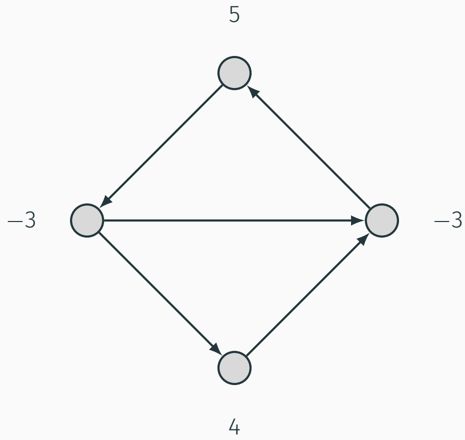
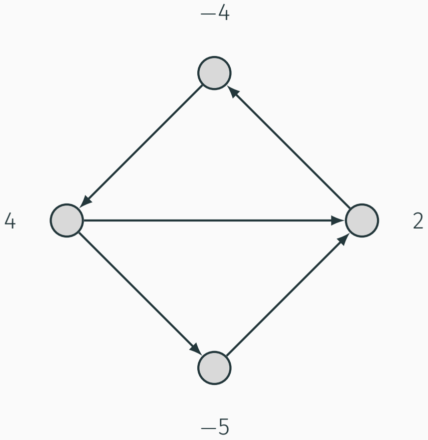
$\mu_{G^i}(x) > 0 \rightarrow x$ is robust infeasible

$\rightarrow$ At most $|V|$ many worst-case scenarios

Visualization Unbalanced Demands



Surplus 3 units



Deficit 3 units

Worst-Case Scenarios: Flow Bounds

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Maximum arc flow in U

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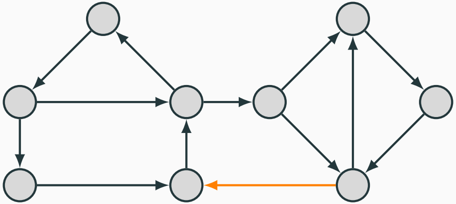

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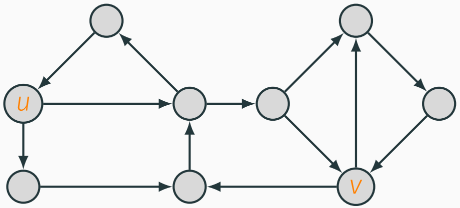
$\rightarrow$ At most $2|A(x)|$ many worst-case flow scenarios for fixed x

Worst-Case Scenarios: Potential Bounds

Maximum potential difference between pair (u, v) of nodes

$$\varphi_{u,v}(x) := \max_{\ell,q,\pi} \quad \pi_u - \pi_v \quad \text{s.t.} \quad \begin{array}{ll} \text{massflow conservation}, & u \in V \\ \text{potential-based flows}, & a = (u,v) \in A \\ \ell \in U, & \text{no bounds} \end{array}$$



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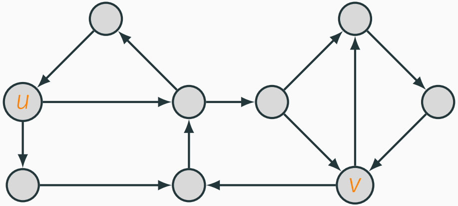

→ At most $|V|^2$ many worst-case scenarios for the potential bounds

Main Result: Characterization of Robust Feasibility

Theorem

Let $x \in X$ be fixed and $G'(x) = (V, A_{ex} \cup \{a \in A_{ca} \colon x_a = 1\})$ be the expanded graph. Let $\mathcal{G}'(x) := \{G^1, \ldots, G^n\}$ with $G^i = (V^i, A^i)$ be the set of connected components of $G'(x)$. Then, expansion x is adjustable robust feasible if and only if

$$\begin{aligned} \mu_{G^i}(x) &= 0 \quad && \text{for all} \quad G^i \in \mathcal{G}'(x) \\ \varphi_{u,v}(x) &\leq \pi_u^+ - \pi_v^- \quad && \text{for all} \quad (u,v) \in (V^i)^2,\ G^i \in \mathcal{G}'(x) \\ \underline{q}_a(x) &\geq q_a^- \quad && \text{for all} \quad a \in A^i,\ G^i \in \mathcal{G}'(x) \\ \bar{q}_a(x) &\leq q_a^+ \quad && \text{for all} \quad a \in A^i,\ G^i \in \mathcal{G}'(x) \end{aligned}$$

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\bar{q}_a(x) &\le q_a^+ && \text{for all} \quad a \in A^i,\ G^i \in \mathcal{G}'(x)
\end{aligned}
$$

$\rightarrow$ At most $|V| + |V|^2 + 2|A|$ many “worst-case” scenarios

Result holds for general compact uncertainty sets U

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Determine a set of finitely many scenarios $S \subseteq U$

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Variant: Add at most one violating scenario per iteration

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Theorem

Algorithm terminates after a finite number of iterations with a global optimal solution or proves infeasibility.

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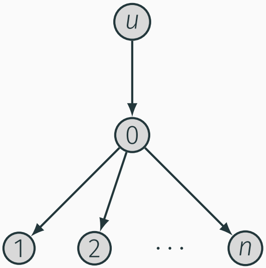

- Source u
- Inner node 0
- Sinks $1, \dots, n$

$$\varphi_a(q_a) = \Lambda_a q_a \left|q_a\right|$$

$$\Lambda_a = 1$$

$$[\pi_w^-, \pi_w^+] = [1, 5]$$

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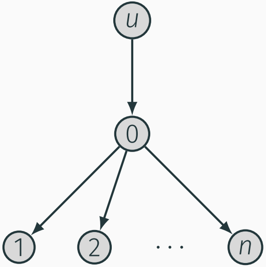

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- Source u
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- Sinks $1, \ldots, n$
- Parallel expansion candidates
- Box uncertainty set

$$U = \{\ell_w \in [0,2], w \in V,\ \ell_0 = 0\} \cap \left\{ \ell_u = \sum_{v \in V_-} \ell_v \right\}$$

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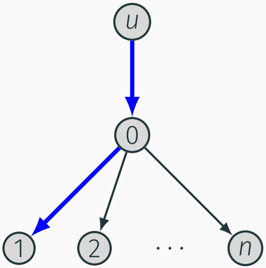

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  - Worst-Case demand: $d_u = d_1 = 2$, remaining nodes demand 0

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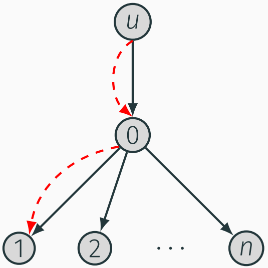

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  - Expansion decision $x_{u,1} = x_{0,1} = 1$

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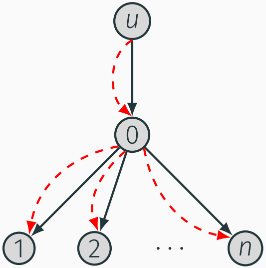

- After n iterations
- $|V_+| \times |V_-|$ worst-case scenarios

$$S = \{\ell_u = \ell_v = 2,\ \ell_w = 0, w \in V_- \setminus \{v\} \text{ for all } v \in V_-\}$$

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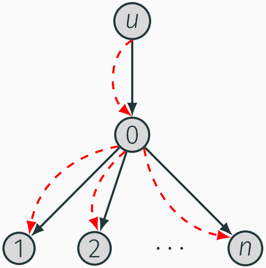

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Why do we need "so many" worst-case scenarios?
$\rightarrow$ Limited supply capacity of the source

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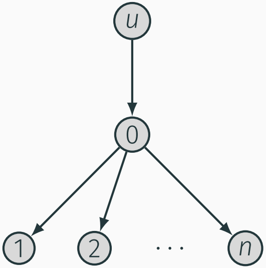

- Same network with larger supply capacity

$$\tilde{U} = \{\ell_v \in [0,2], v \in V_-,\ \ell_0 = 0,\ \ell_u \leq 2\,|V_-|\}$$
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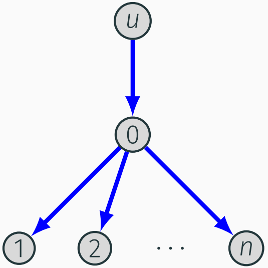

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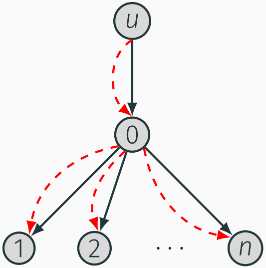

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- Algorithm terminates after a single iteration

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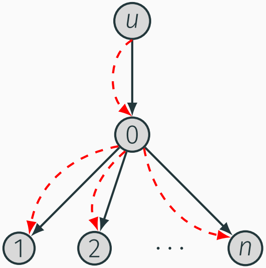

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- First iteration
- Worst-case scenario $d_u = 2|V_-|$, $d_i = 2, i \in [n]$
- Algorithm terminates after a single iteration

- Real-world networks: sources can supply many sinks
  $\rightarrow$ very few worst-case scenarios in practice

Computational Results

Computational Setup

Implemented in Python 3.7 and Pyomo 6.7.0

Solving MINLPs with Gurobi 10.0.3

Time limit of 24 hours per instance

Gas networks $\varphi_a = \Lambda_a q_a |q_a|$

Expansion candidates are in parallel with up to four different diameters

instance	#nodes	#sources	#sinks	#pipes	#short pipes
GasLib-40	40	3	29	39	6
GasLib-60	60	3	39	61	18

Computational Results

Consider four different polyhedral uncertainty sets
$\rightarrow$ with and without correlations between sinks

Add to the plain algorithm

- Acyclic inequalities (Habeck and Pfetsch 2022)
- Mixed-integer convex relaxation $\rightarrow$ lower bounds for the MINLPs

$\rightarrow$ only used for computing lower bounds

General approach is exact

Robustifying Existing Networks

Robustifying Existing Networks

Plain Approach (Left: GasLib-40, Right: GasLib-60)

#Solved	4 of 4		
	Min	Median	Max
#Scenarios	1	2	2
Runtime (s)	807.65	1395.33	1578.68

#Solved	4 of 4		
	Min	Median	Max
#Scenarios	1	1	1
Runtime (s)	1117.37	1175.83	3009.57

Approach with lower bound strengthening

#Solved	4 of 4		
	Min	Median	Max
#Scenarios	1	2	2
Runtime (s)	332.21	1149.98	2042.90

#Solved	4 of 4		
	Min	Median	Max
#Scenarios	1	1	1
Runtime (s)	564.06	995.62	1037.74

Greenfield Approach

Greenfield Approach

Plain Approach (Left: GasLib-40, Right: GasLib-60)

#Solved	1 of 4		
	Min	Median	Max
#Scenarios	1	1	1
Runtime (s)	7320.85	7320.85	7320.85

#Solved	1 of 4		
	Min	Median	Max
#Scenarios	2	2	2
Runtime (s)	81 895.84	81 895.84	81 895.84

Approach with lower bound strengthening

#Solved	3 of 4		
	Min	Median	Max
#Scenarios	1	3	3
Runtime (s)	4066.79	39 963.87	50 183.53

#Solved	1 of 4		
	Min	Median	Max
#Scenarios	2	2	2
Runtime (s)	51 290.35	51 290.35	51 290.35

Summary and Outlook

An algorithm to compute adjustable robust network designs for nonlinear flows

- Finitely many “worst-case scenarios”
- Finite termination for arbitrary compact uncertainty sets
- Approach performs well in practice

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Future research

- Extension to active elements
- Valid inequalities for network design problems with potential-based flows

Main Source

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Adjustable robust nonlinear network design without controllable elements under load scenario uncertainties

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