Dantzig-Wolfe and Fenchel decompositions : hybridation and degeneracy

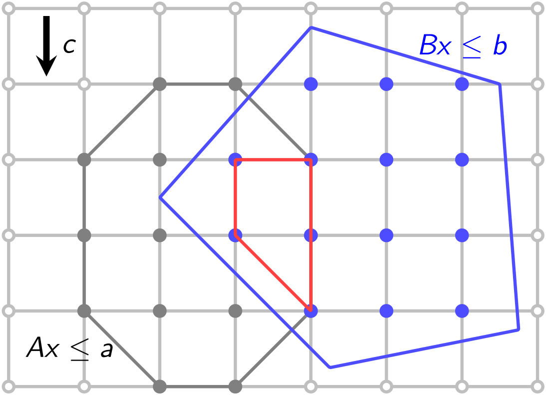
François Lamothe, Alain Hait, Emmanuel Rachelson, Claudio Contardo, Bernard Gendron

2 juillet 2025

Introduction

$$\begin{aligned} \min_{x \in \mathbb{Z}} \ & cx \\ \text{s.t. } & Ax \leq a \\ & \color{blue}{Bx \leq b} \end{aligned}$$

$$\min_{x \in \mathbb{Z}} \; cx$$
$$\text{s.t. } Ax \leq a$$
$$Bx \leq b$$



Introduction

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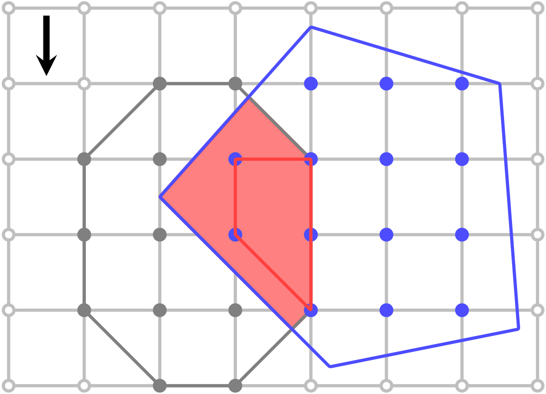
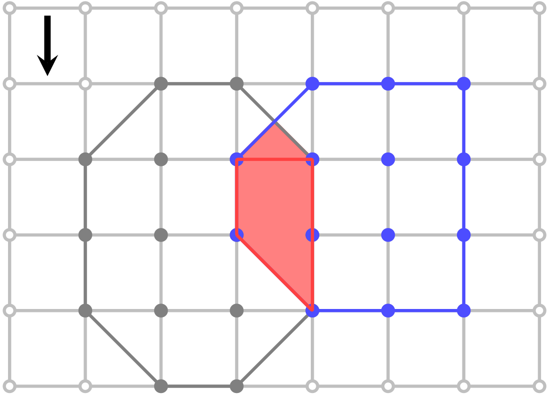


Linear relaxation

$$\min_{x \in \mathbb{Z}} \; cx$$
$$\text{s.t. } Ax \leq a$$
$$x \in P$$

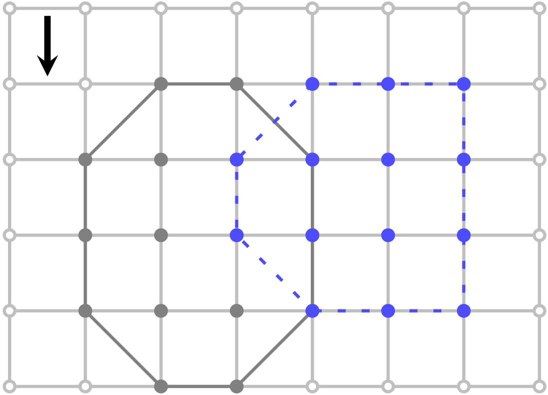


A better relaxation

Introduction

$$\min_{x \in \mathbb{Z}} \; cx$$
$$\text{s.t. } Ax \leq a$$
$$x \in P$$

$$\max \; \pi x$$
$$\text{s.t. } x \in P$$



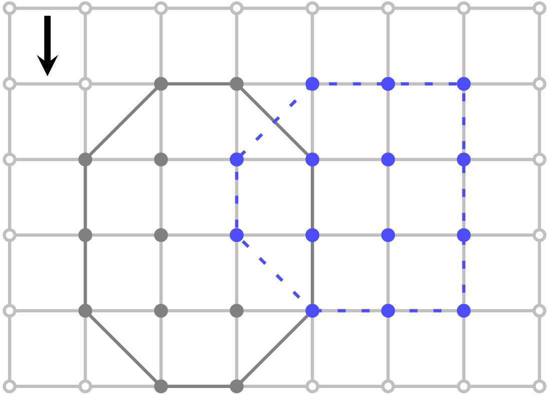
Polyhedron known only implicitly : oracle

Summary

- Dantzig-Wolfe decomposition
- Fenchel decomposition
- Proposed hybridation
- An illustration of degeneracy

Dantzig Wolfe decomposition

$$\min_{x \in \mathbb{R}} \; cx$$
$$\text{s.t. } Ax \leq a$$
$$x \in P$$



Dantzig Wolfe decomposition

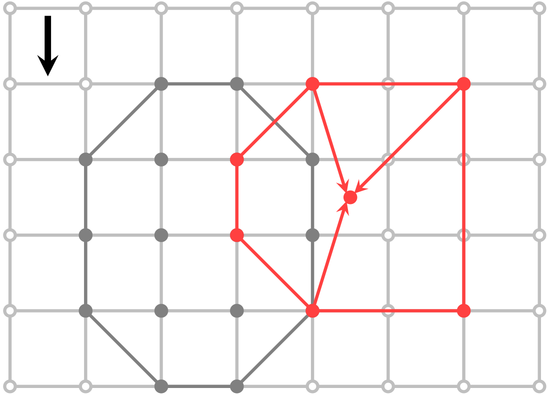
$$\min_{x \in \mathbb{R}} cx$$

$$\text{s.t. } Ax \leq a$$

$$x = \sum \lambda_i x_i$$

$$\sum \lambda_i = 1$$

$$\lambda_i \geq 0$$



Point = convex combination of vertices

Dantzig Wolfe decomposition

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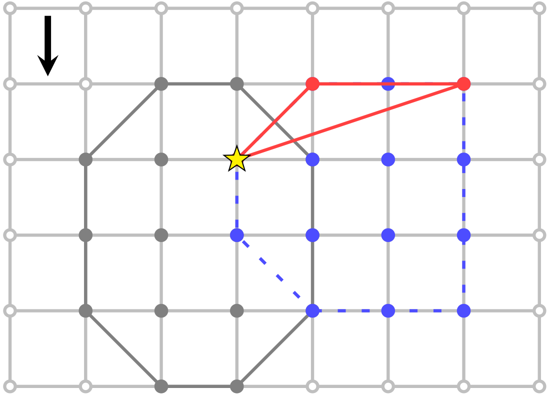

Start with few vertices

Dantzig Wolfe decomposition

$$\min_{x \in \mathbb{R}} cx$$

s.t. $Ax \leq a$

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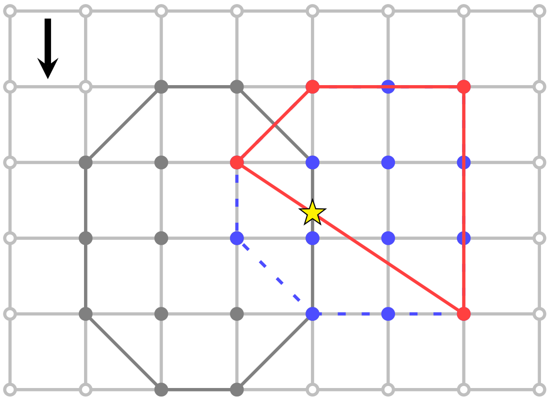

Add new vertex $\rightarrow$ better inner approx

Dantzig Wolfe decomposition

$$\min_{x \in \mathbb{R}} \; cx$$
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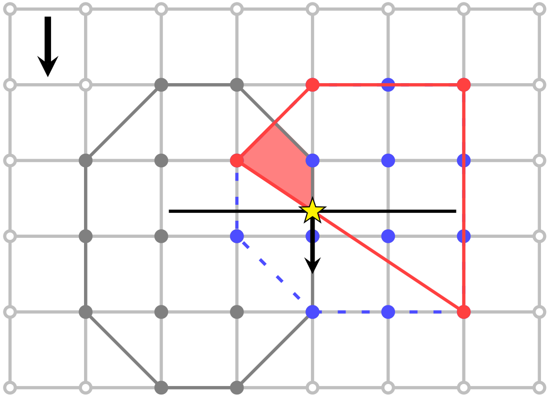

Dual vars of master pb $\rightarrow$ certify solution is optimal

Dantzig Wolfe decomposition

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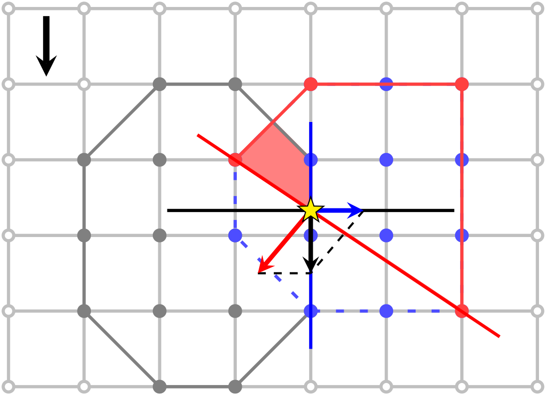

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Dantzig Wolfe decomposition

$$\min_{x \in \mathbb{R}} cx$$

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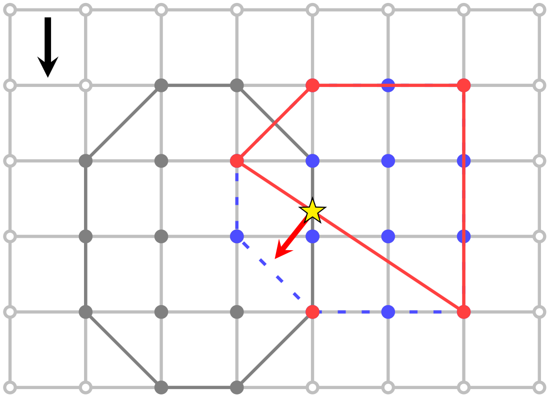
$\lambda_i \geq 0$

$\max \pi x$

s.t. $x \in P$



Dual direction + oracle $\rightarrow$ new vertex

$$\min_{x \in \mathbb{R}} cx$$

$$\text{s.t. } Ax \leq a$$

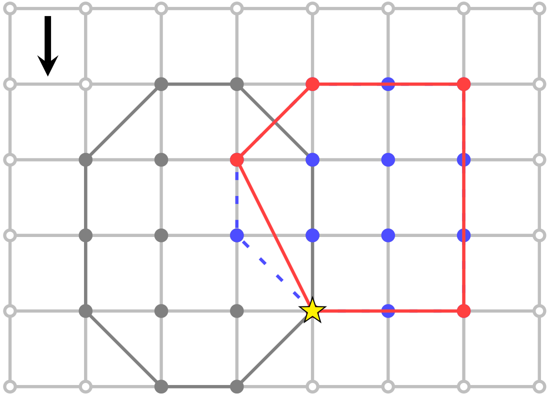
$$x = \sum \lambda_i x_i$$

$$\sum \lambda_i = 1$$

$$\lambda_i \geq 0$$

$$\max \pi x$$

$$\text{s.t. } x \in P$$

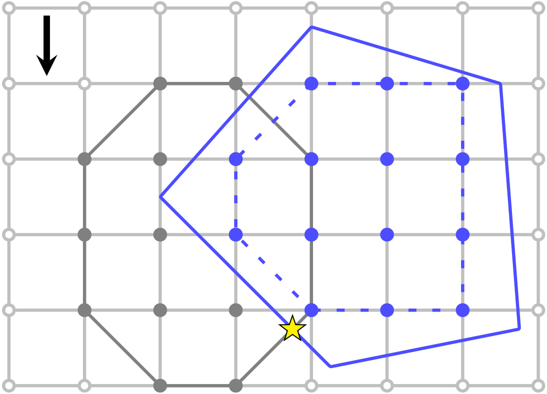


Repeat until optimal

Summary

- Dantzig-Wolfe decomposition
- Fenchel decomposition
- Hybridation
- Degeneracy

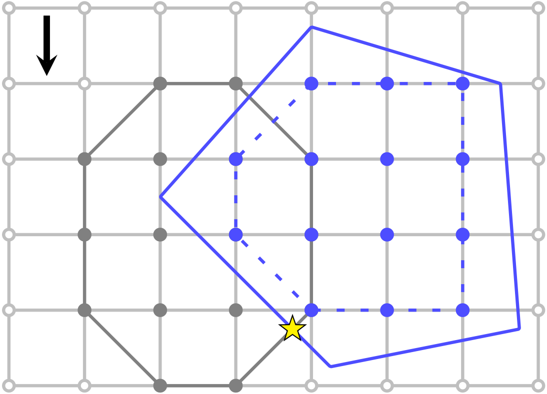
Fenchel decomposition



Fenchel decomposition = outer approximation

Fenchel decomposition

$$\min_{x \in \mathbb{R}} \; cx$$
$$\text{s.t. } Ax \leq a$$
$$\pi^j x \leq \pi^j_0 \quad \forall j \in J$$



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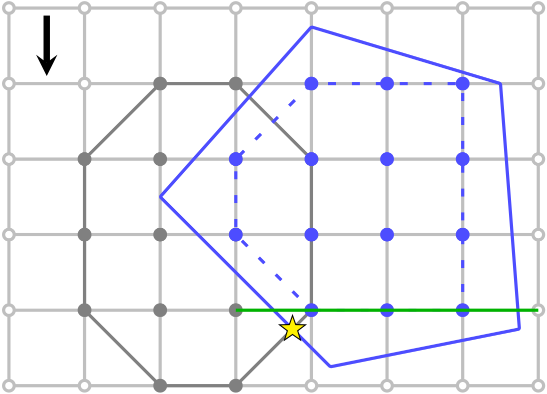


New cuts generated with separation problem

Fenchel decomposition

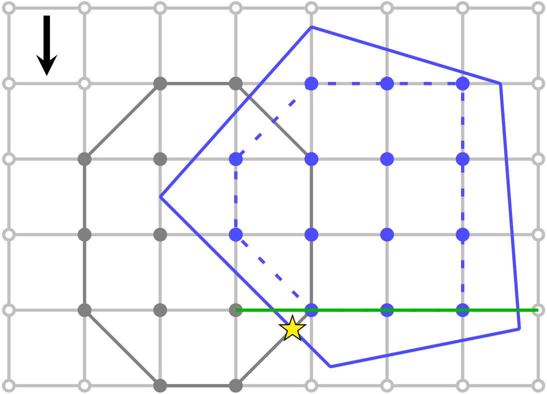
$$\min_{x \in \mathbb{R}} \; cx$$

$$\text{s.t. } Ax \leq a$$

$$\pi^j x \leq \pi_0^j \quad \forall j \in J$$

$$\max_{\pi,\ \pi_0} \; \pi x^* - \pi_0$$

$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$



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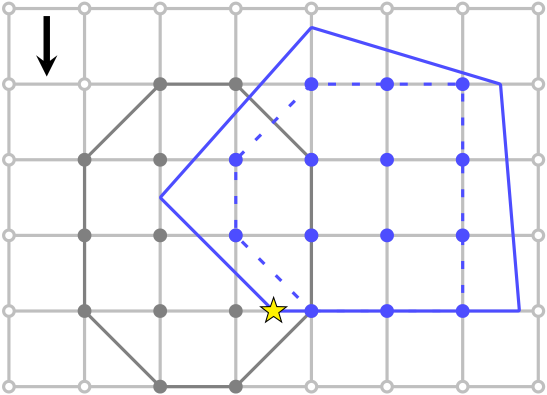

Adding cut $\rightarrow$ tigher outer approx

Fenchel decomposition

$$\min_{x \in \mathbb{R}} \; cx$$
$$\text{s.t. } Ax \leq a$$
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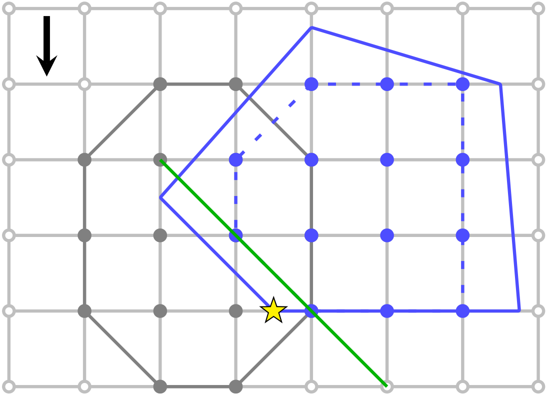

Repeat until optimal

Fenchel decomposition

$$\min_{x \in \mathbb{R}} \; cx$$

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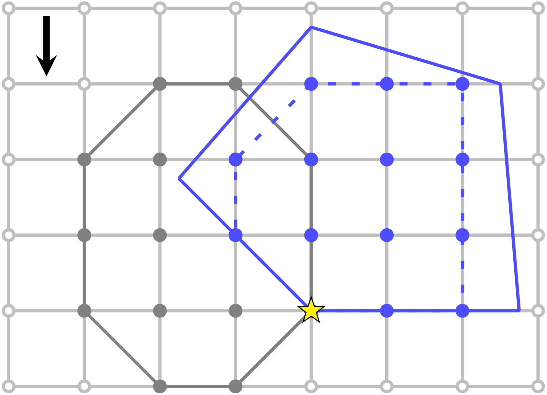


Repeat until optimal

Normalization

$$\max_{\pi,\ \pi_0} \ \pi x^* - \pi_0$$
$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$

Normalization

$$\max_{\pi,\ \pi_0} \pi x^* - \pi_0$$

$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$

$$\max_{\lambda_i} 0$$

$$x^* = \sum \lambda_i x_i$$

$$\sum \lambda_i = 1$$

$$\lambda_i \geq 0$$

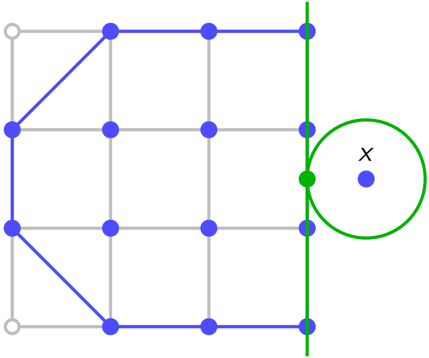
Normalization

$$\max_{\pi,\ \pi_0} \pi x^* - \pi_0$$
$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$

Model is unbounded

Normalization

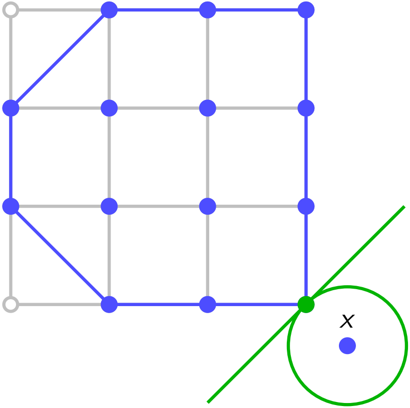
$$\max_{\pi,\ \pi_0} \pi x^* - \pi_0$$
$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$
$$||\pi||_2 \leq 1$$



Normalisation with L2 norm

Normalization

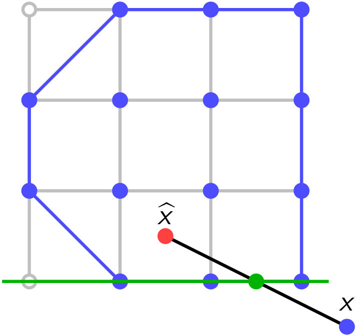
$$\max_{\pi,\ \pi_0} \pi x^* - \pi_0$$
$$\text{s.t. } \pi x_i \leq \pi_0 \quad \forall i \in I$$
$$||\pi||_2 \leq 1$$



No guarantee to get a facet

Normalization

$$\max_{\pi,\ \pi_0} \ \pi x^* - \pi_0$$
$$\text{s.t. } \pi x_i \le \pi_0 \quad \forall i \in I$$
$$|\pi(\hat{x} - x)| \le 1$$

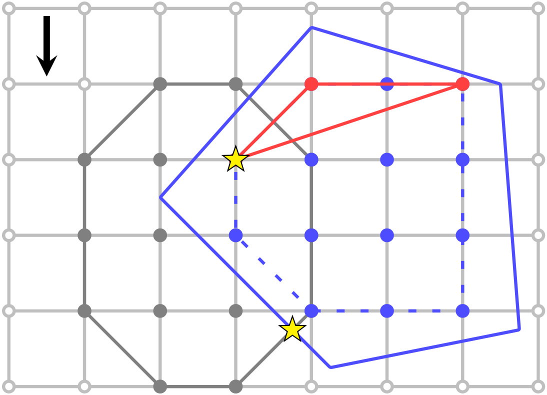


Directional normalisation : always a facet

Summary

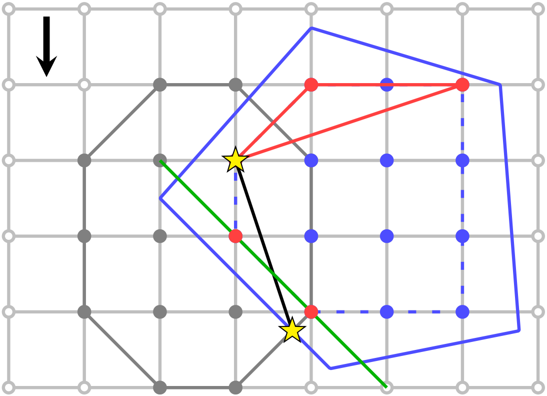
- Dantzig-Wolfe decomposition
- Fenchel decomposition
- Hybridation
- Degeneracy

Hybridation



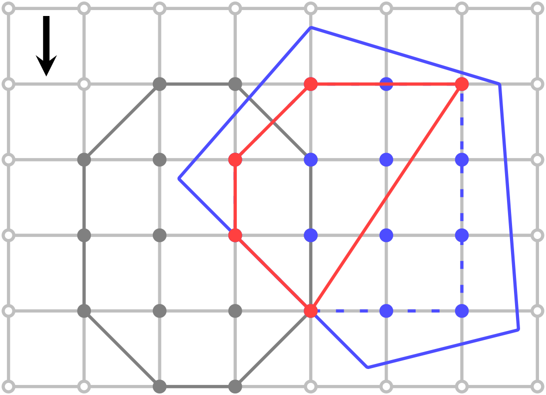
Both inner and outer approx in one algo

Hybridation



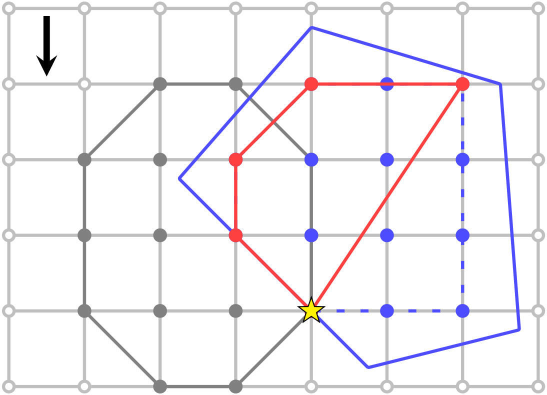
Cut outer solution : directional normalization toward inner sol
Obtain a cut + supporting vertices

Hybridation



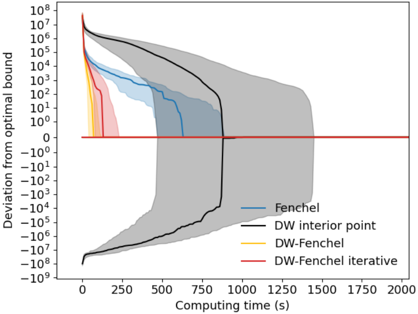
Update both approximations

Hybridation



Repeat until optimal

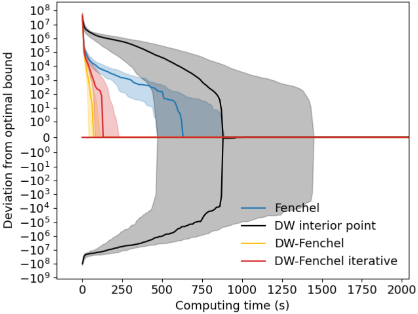
Hybridation



Unsplittable flow problem : root node

Best setup for our method

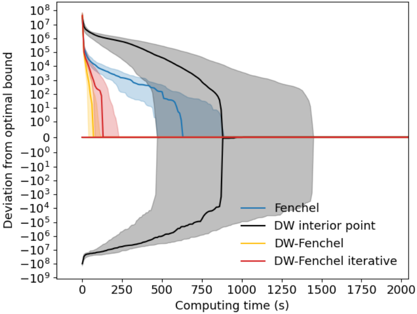
- Many oracle calls $\to$ Fast oracle required
  example : knapsack



Unsplittable flow problem : root node

Best setup for our method

- Many oracle calls $\rightarrow$ Fast oracle required
  example : knapsack
- Symetries/several optimal solution : Fenchel is slow
  Our method suffers less from symetries

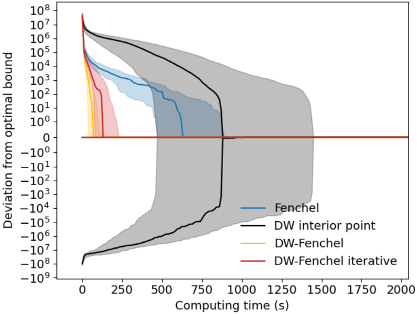


Unsplittable flow problem : root node

Hybridation

Best setup for our method

- Many oracle calls → Fast oracle required
  example : knapsack
- Symetries/several optimal solution : Fenchel is slow
  Our method suffers less from symetries
- Degeneracy :
  Dantzig-Wolfe is slow
  Fenchel methods don't use duals



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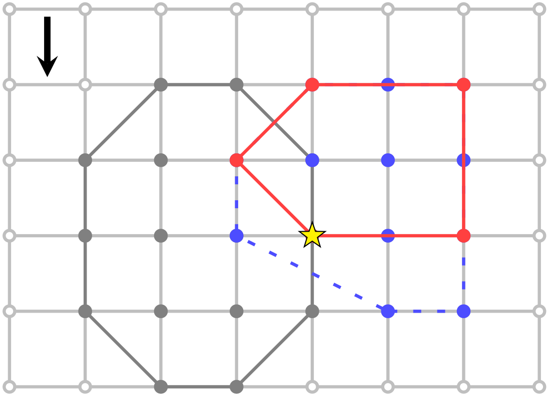
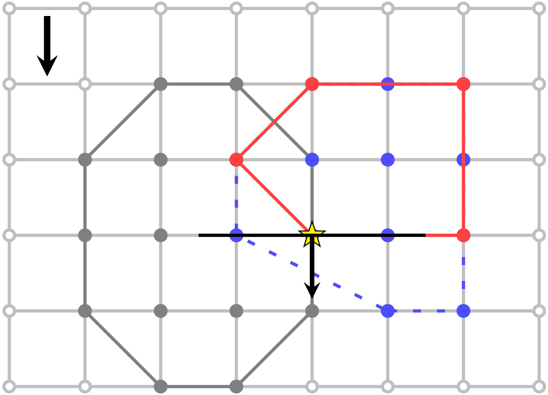
Degeneracy

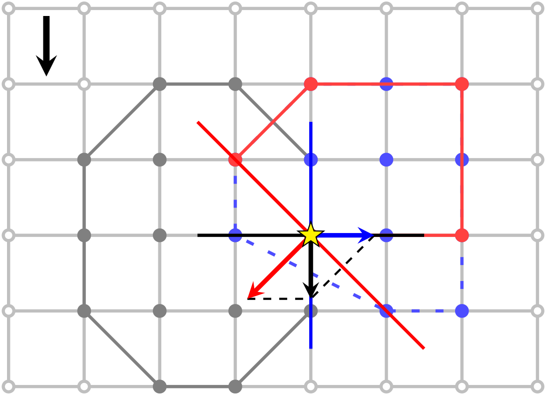


Degeneracy



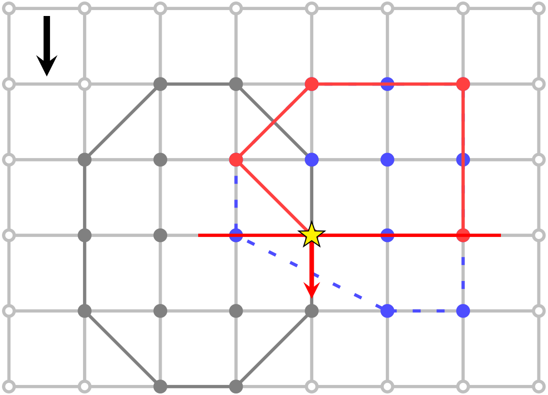
Dual problem = find a bound

Degeneracy



Dual bound decomposes in two parts

Degeneracy



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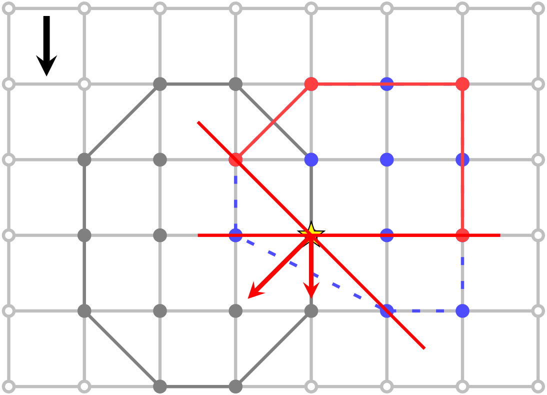
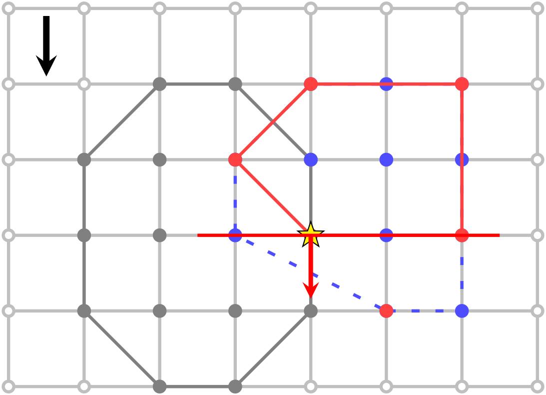


What if there are several decompositions ?

Degeneracy



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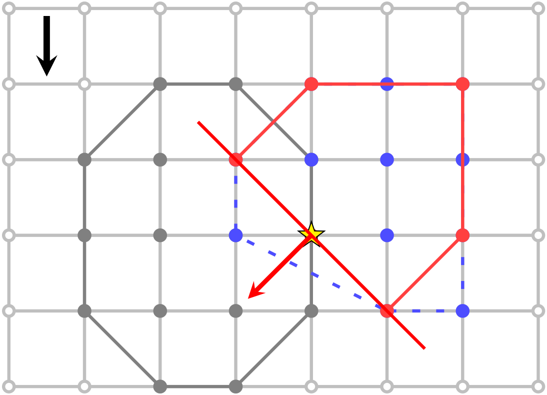

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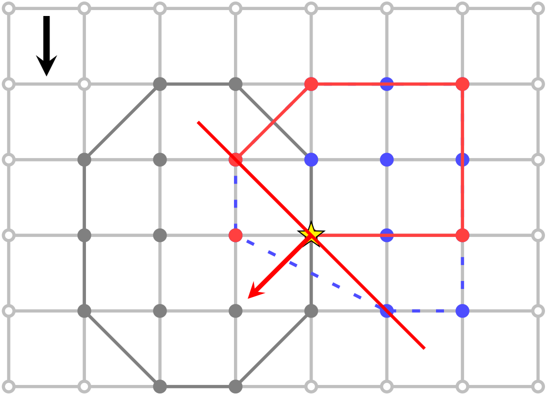

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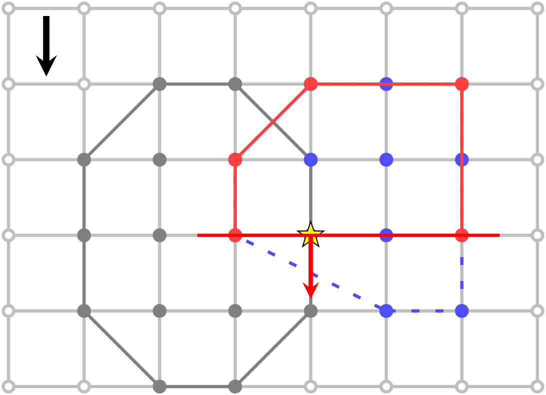

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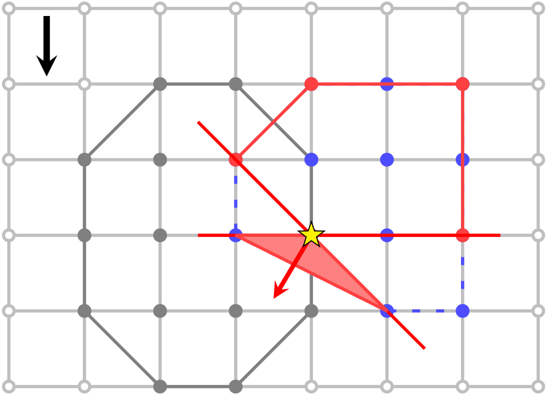

Solving degeneracy ?

Conclusion

Recap :

- Inner approximation : Dantzig-Wolfe
- Outer approximation : Fenchel
- Hybridation can be good ! Use inner and outer approx
- Degeneracy is Dantzig-Wolfe bane
- How to prevent degeneracy in Dantzig-Wolfe ?

Lamothe et al. On the integration of Dantzig-Wolfe and Fenchel decompositions via directional normalizations.
https ://hal.science/hal-04875156v1

Thank you for your attention !

Lamothe et al. On the integration of Dantzig-Wolfe and Fenchel decompositions via directional normalizations.
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