Integer Programs meet Fixed-Parameter Tractability

Alexandra Lassota (TU/e)

Integer Programming

$$Ax \leq b, \; x \in \mathbb{Z}^n$$

Integer Programming

Introduction and Relevance

$$Ax \leq b, \; x \in \mathbb{Z}^n$$

Integer Programming

~~Introduction and Relevance~~

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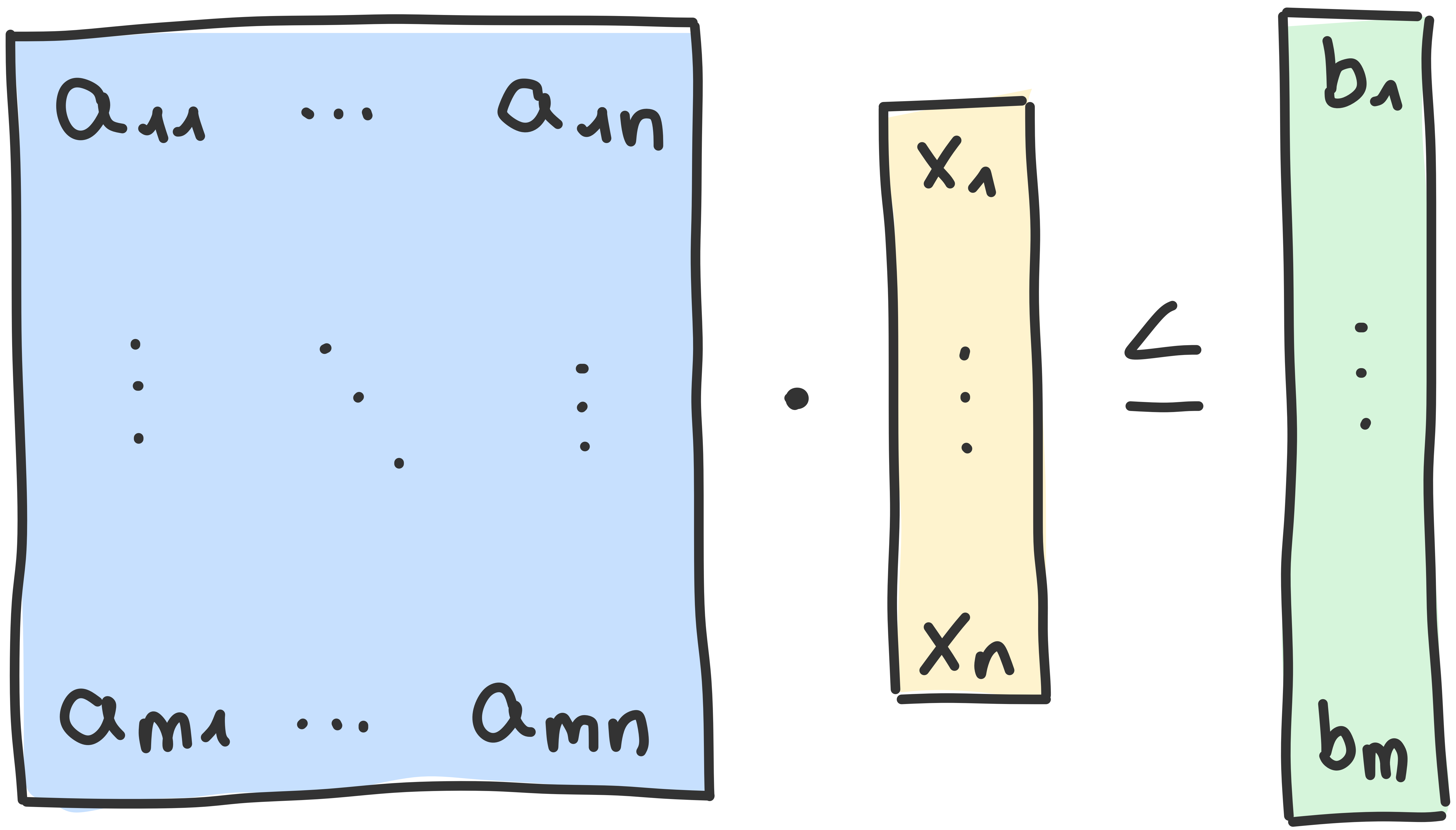
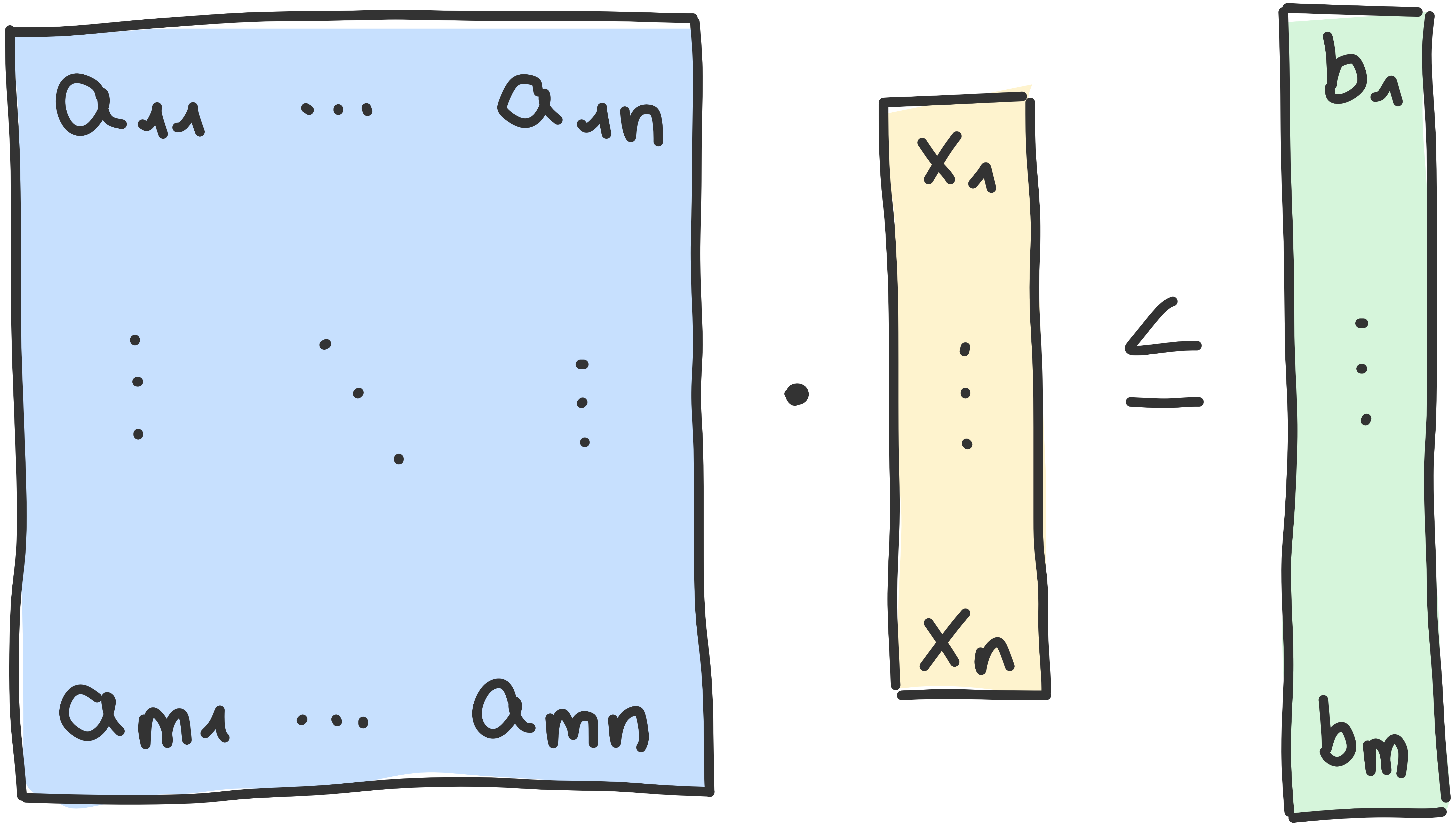
Integer Programming



Integer Programming



NP-hard -> exponential blow-up in runtime

How do we deal the NP-hardness

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- work on solvers

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Fixed-Parameter Tractability

Given: Problem P with parameter k

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If we can solve any instance of P in time

$$f(k) \cdot \mathrm{poly}(|I|)$$

then P is in FPT/ P is fixed-parameter tractable w.r.t. k

Fixed-Parameter Tractability - Example

Given: $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, $x \in \{0,1\}^n$ with parameter n

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Instance size: $O(nm \log(m\Delta))$

$\uparrow \|A\|_\infty$

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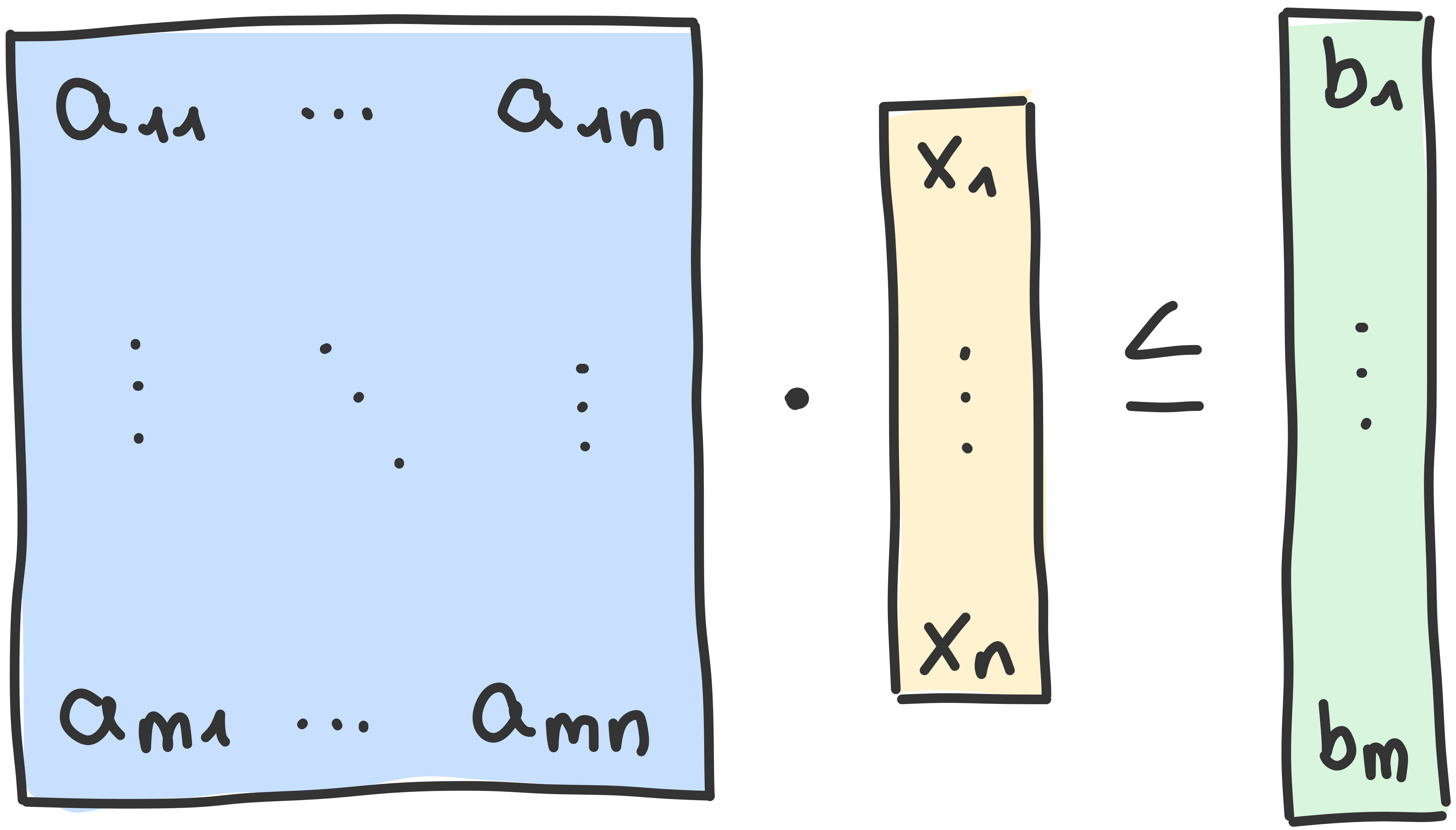
Algorithm: Guess and test solution; $\underbrace{2^n}_{f(n)} \cdot \underbrace{O(nm)}_{poly(|I|)}$

Integer Programming meets FPT



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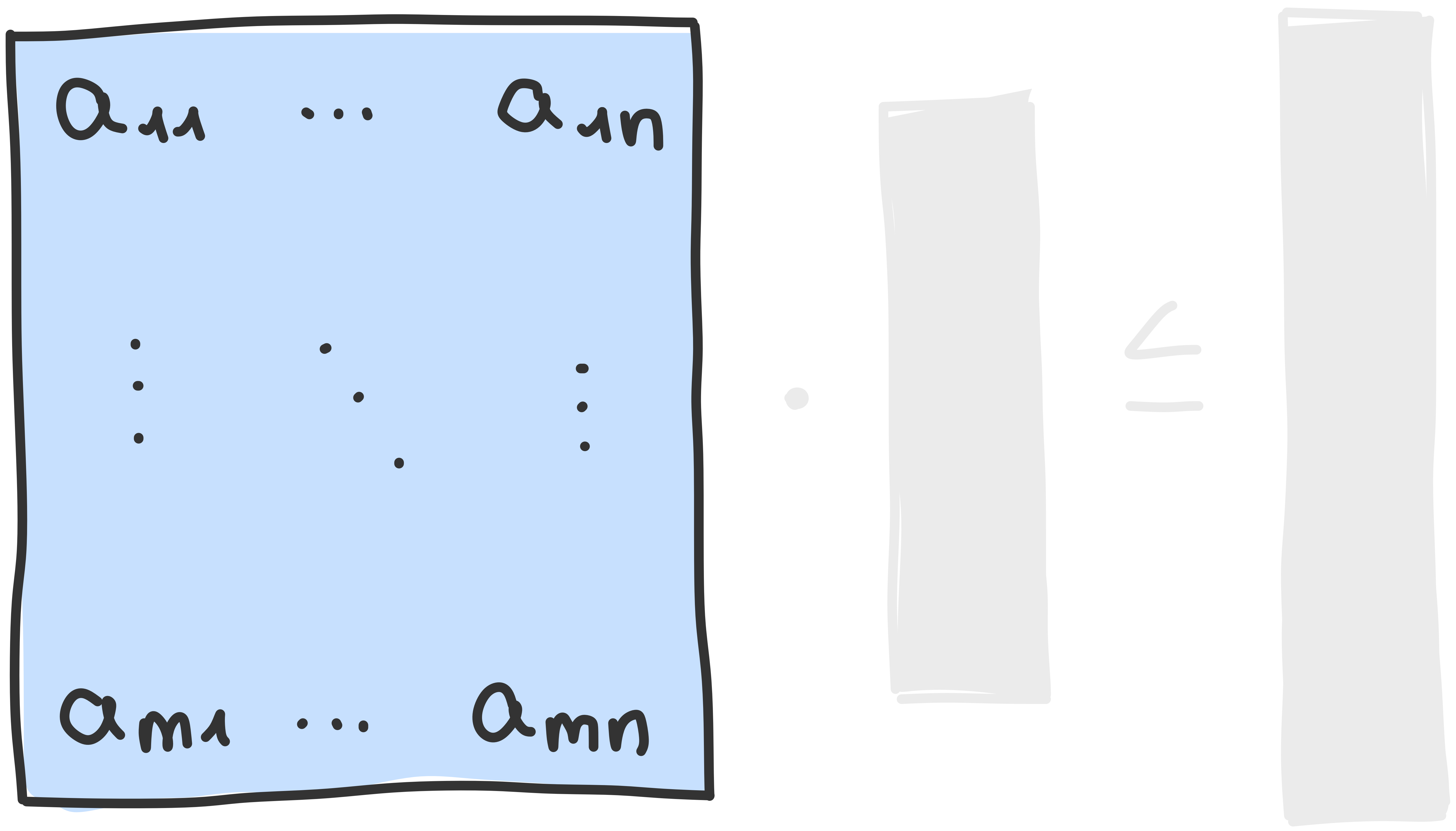

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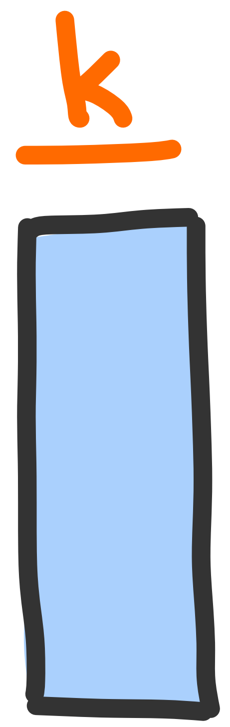

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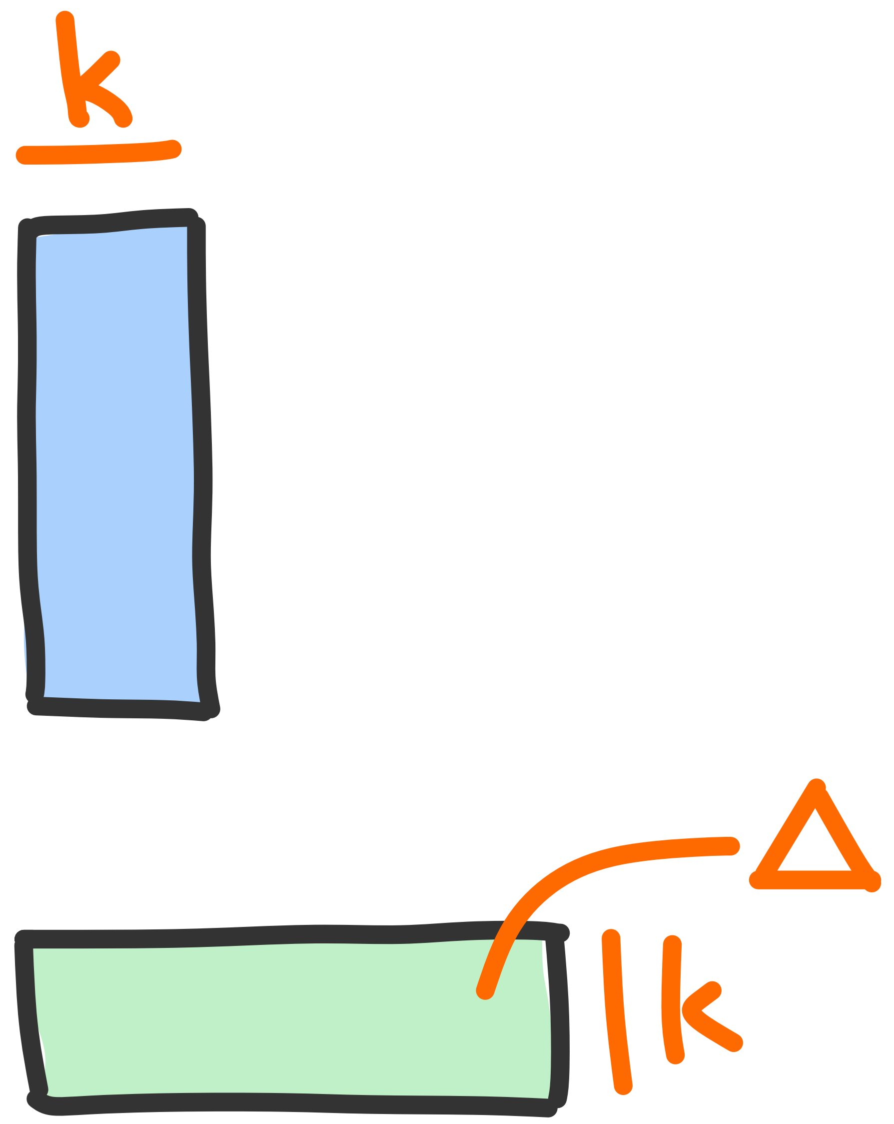

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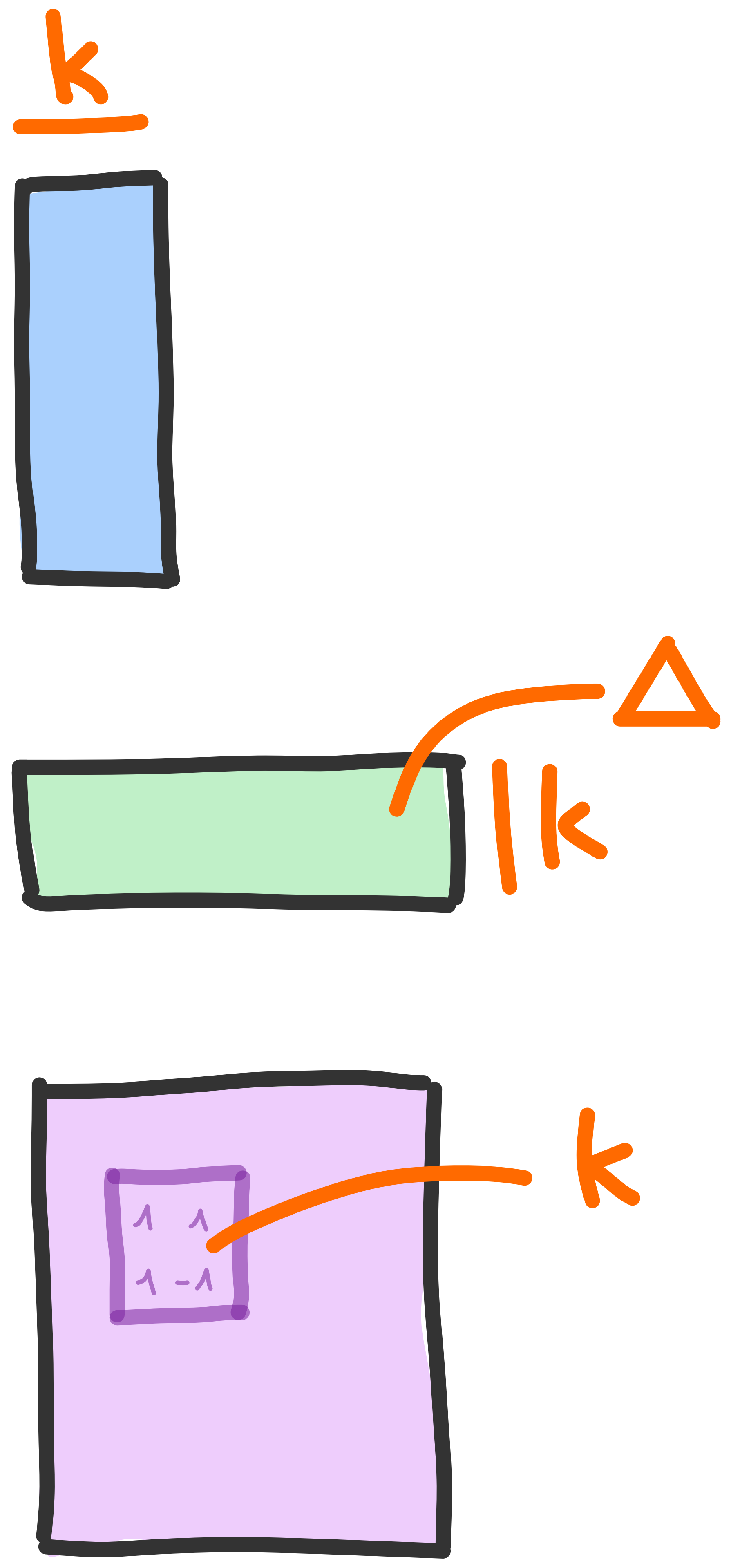

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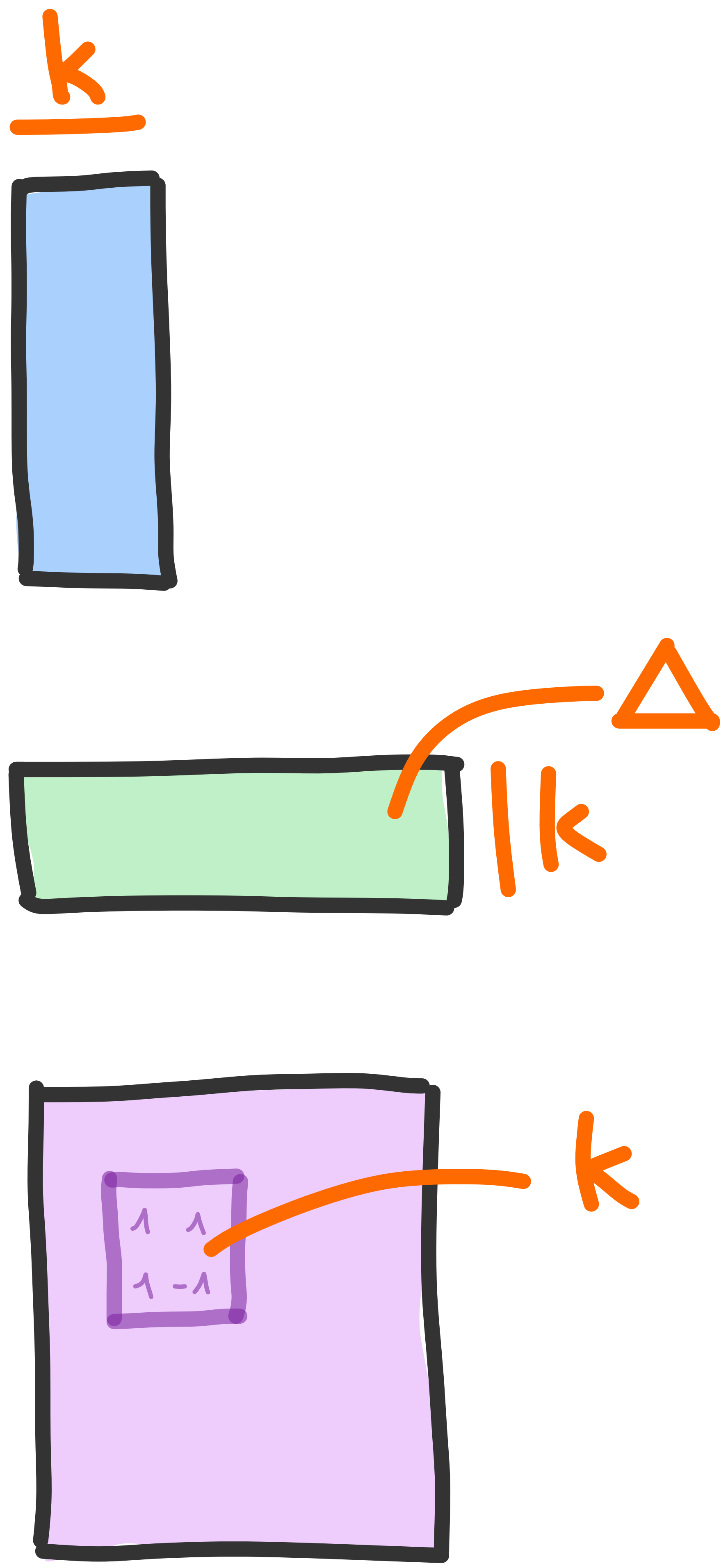

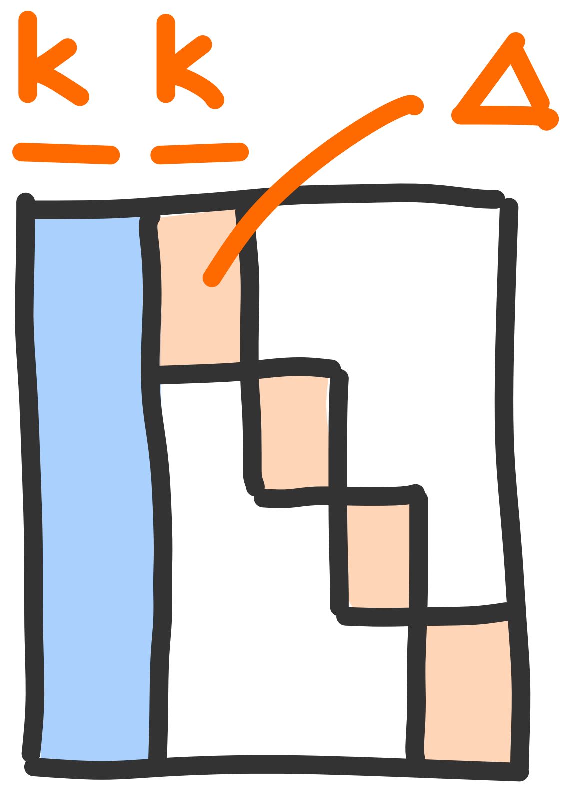

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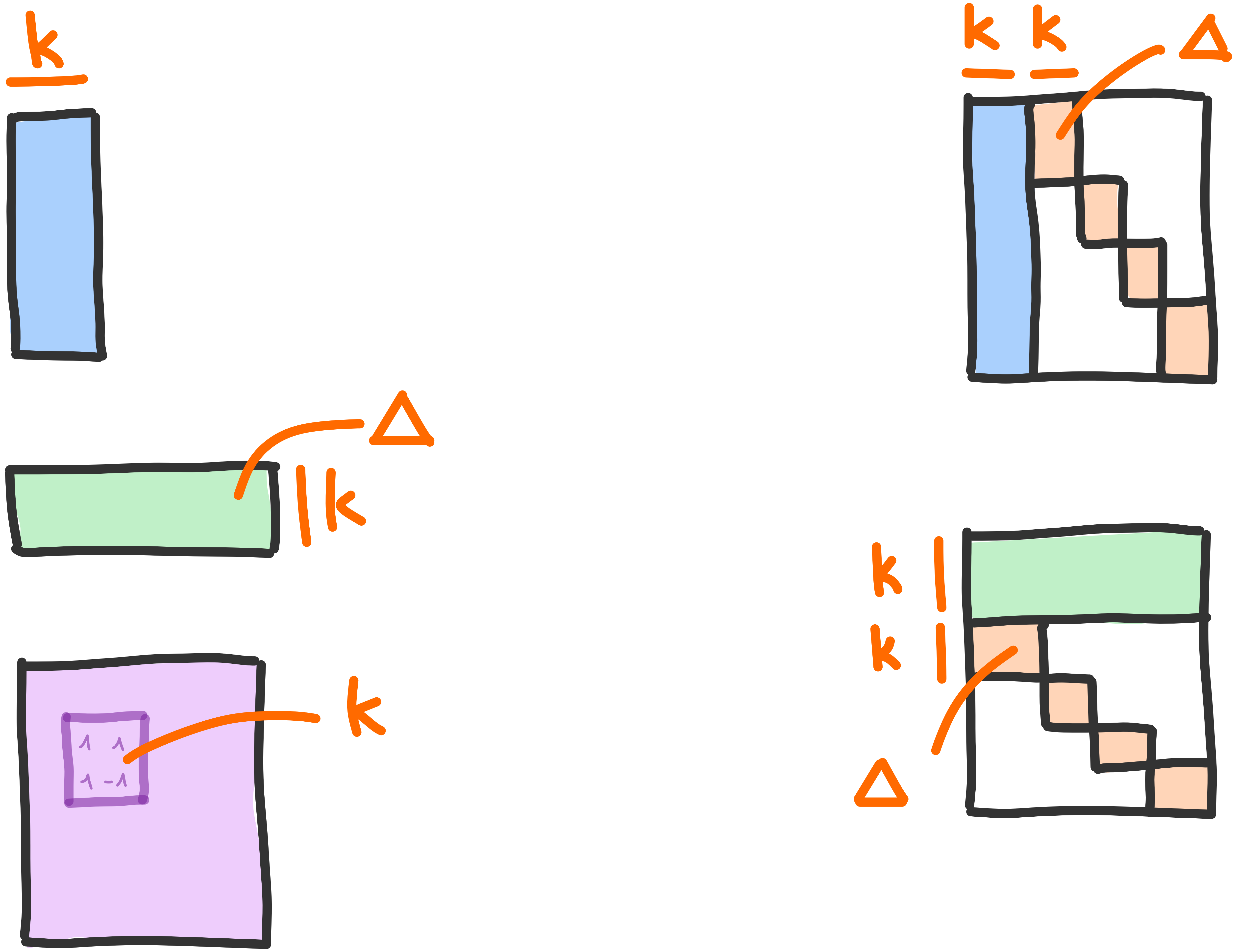

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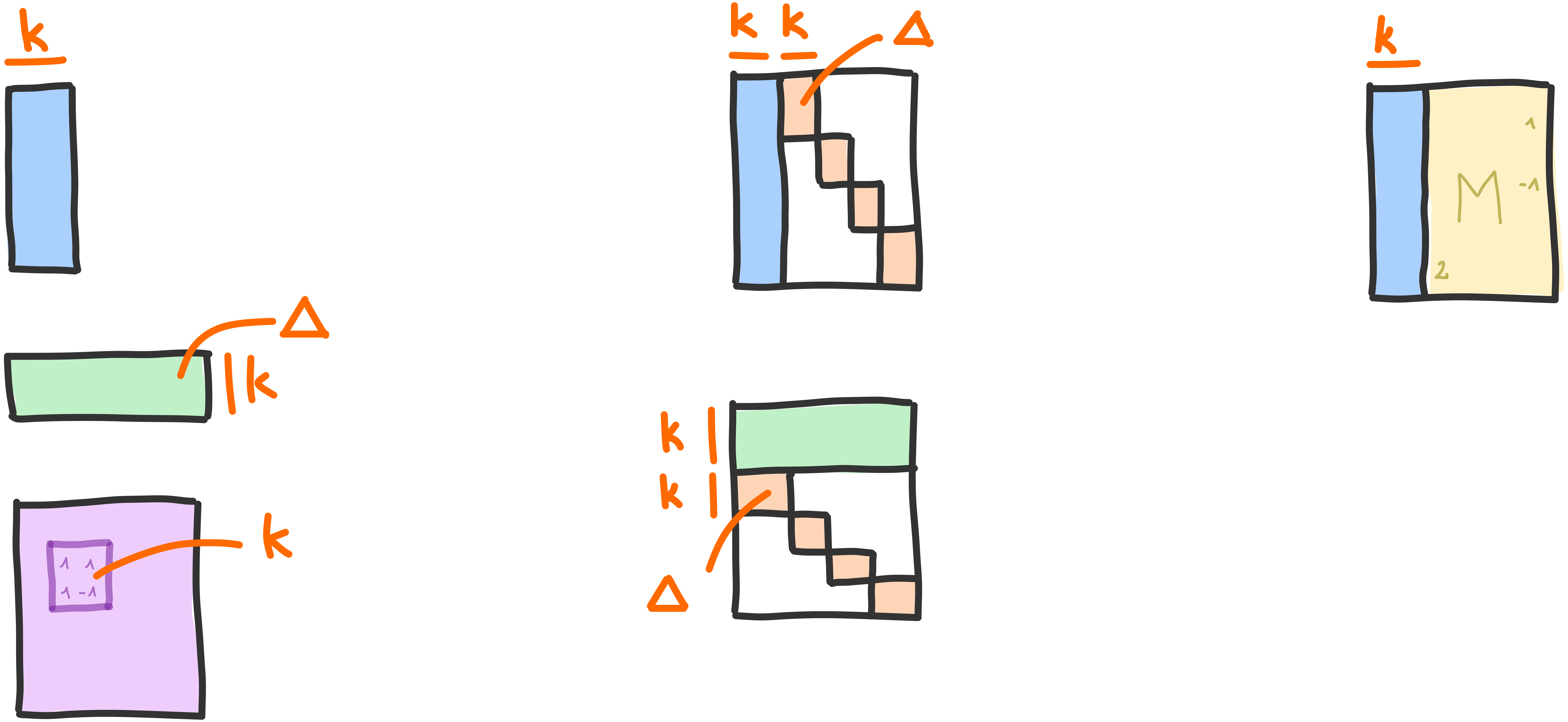

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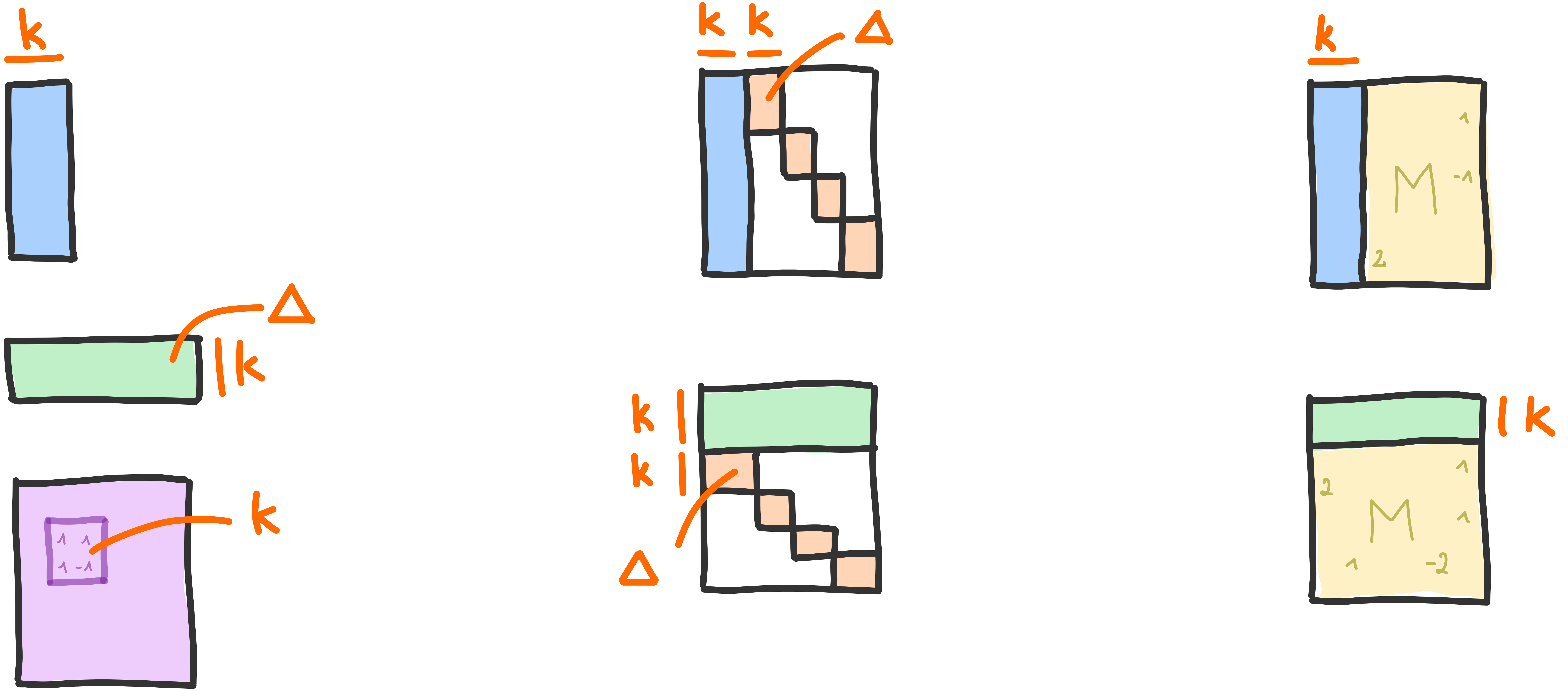

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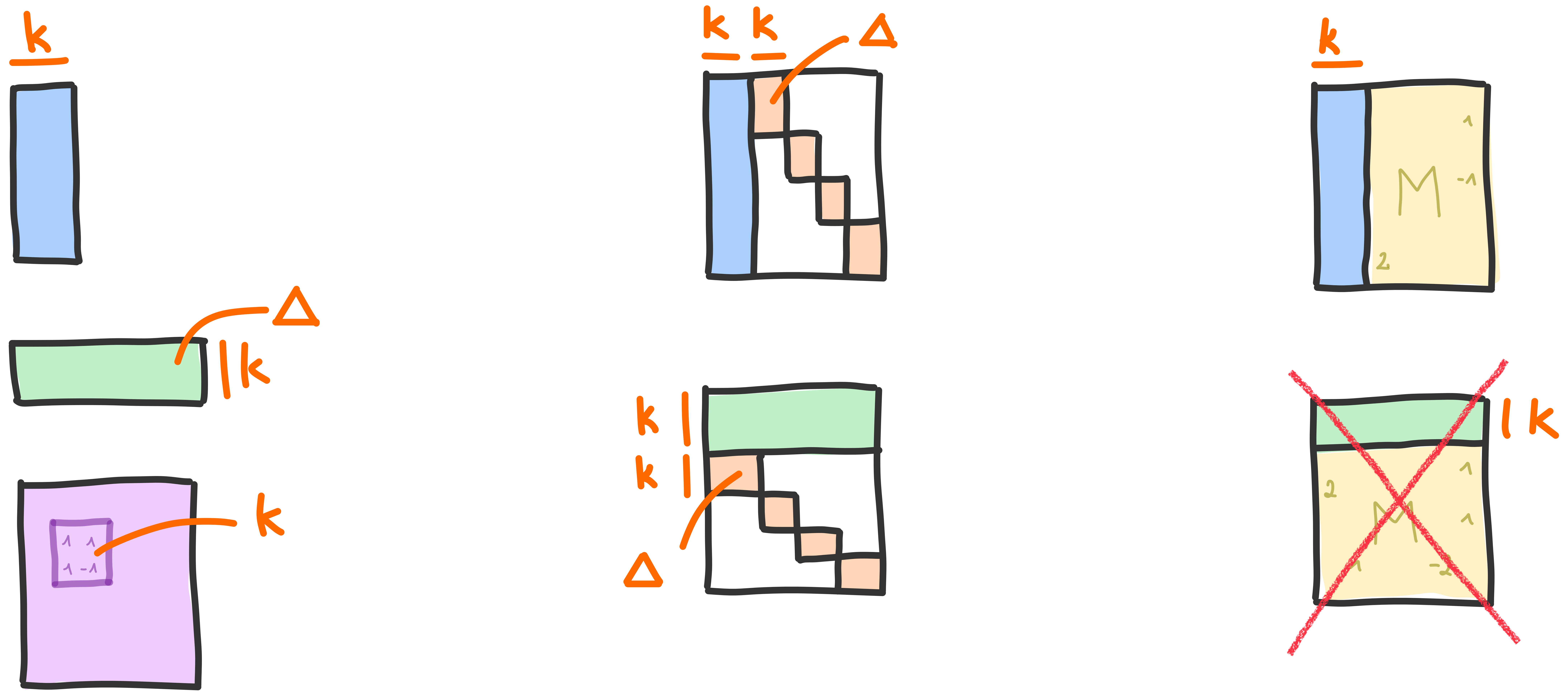

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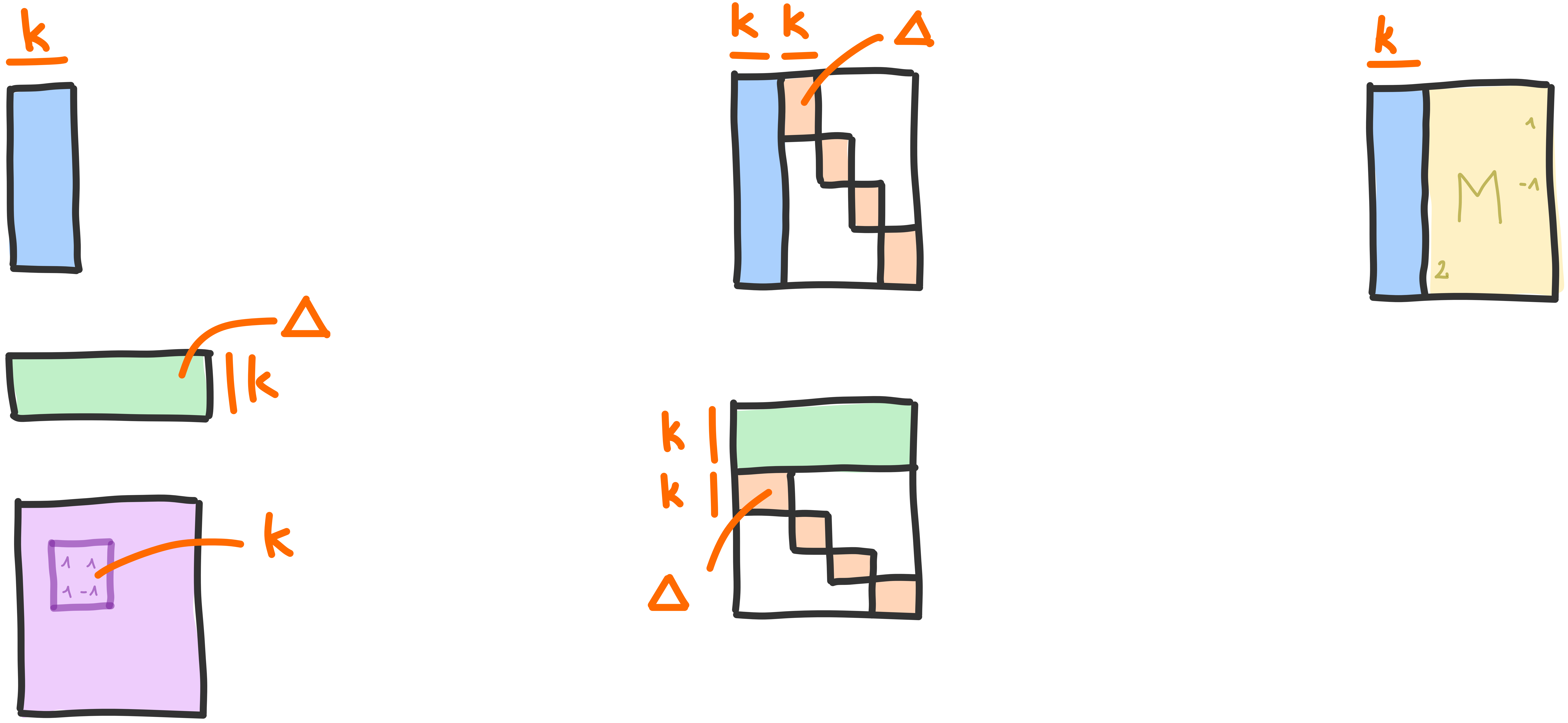
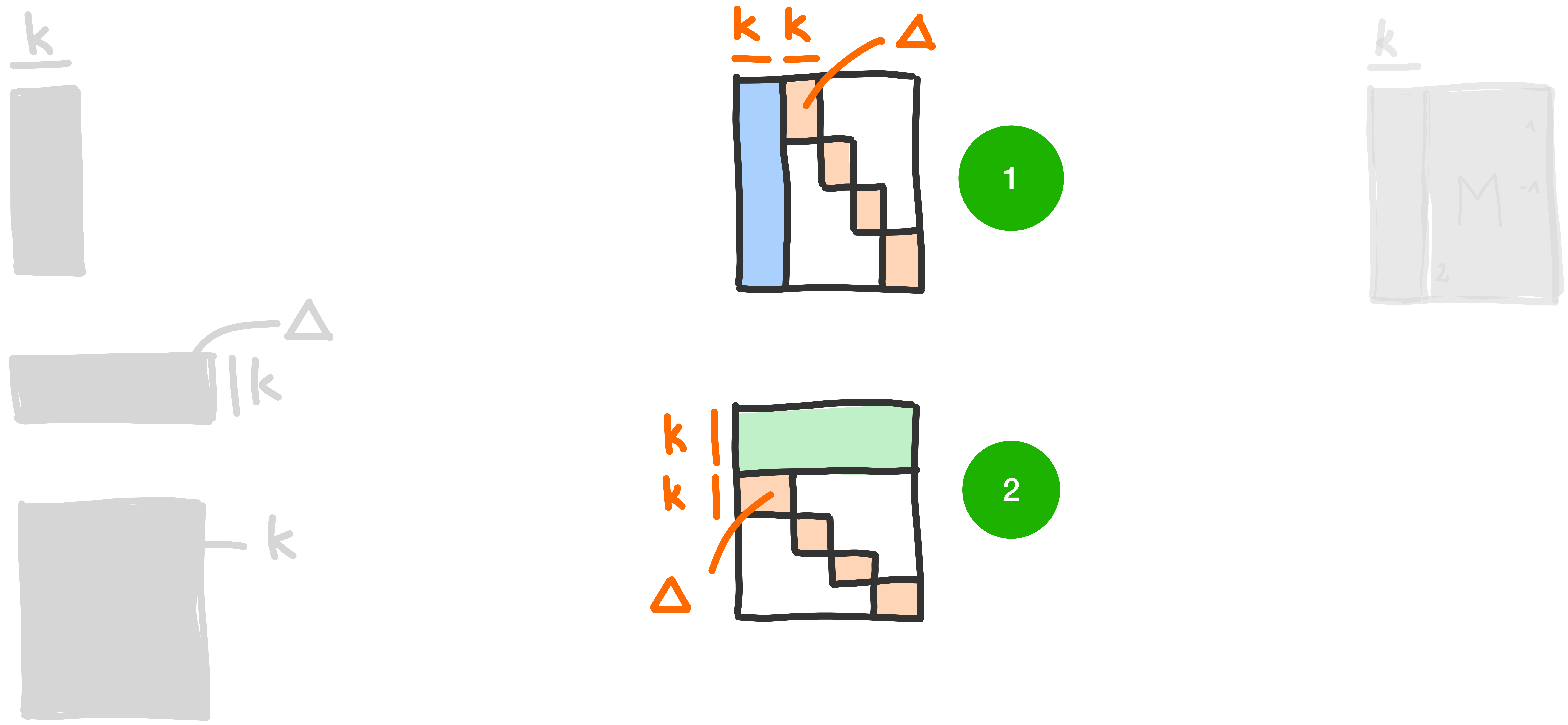


Integer Programming meets FPT - Agenda



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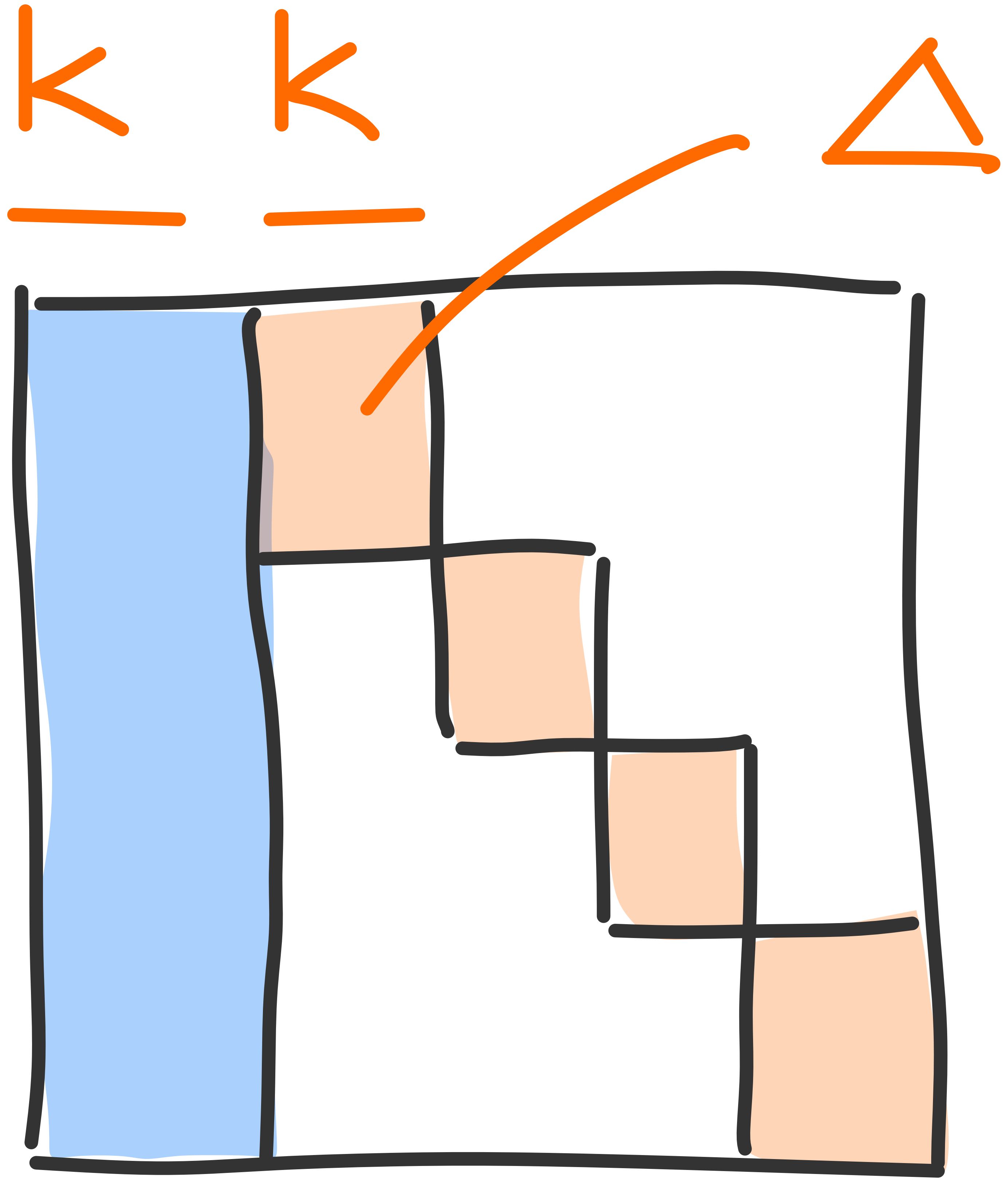

2-stage stochastic IPs

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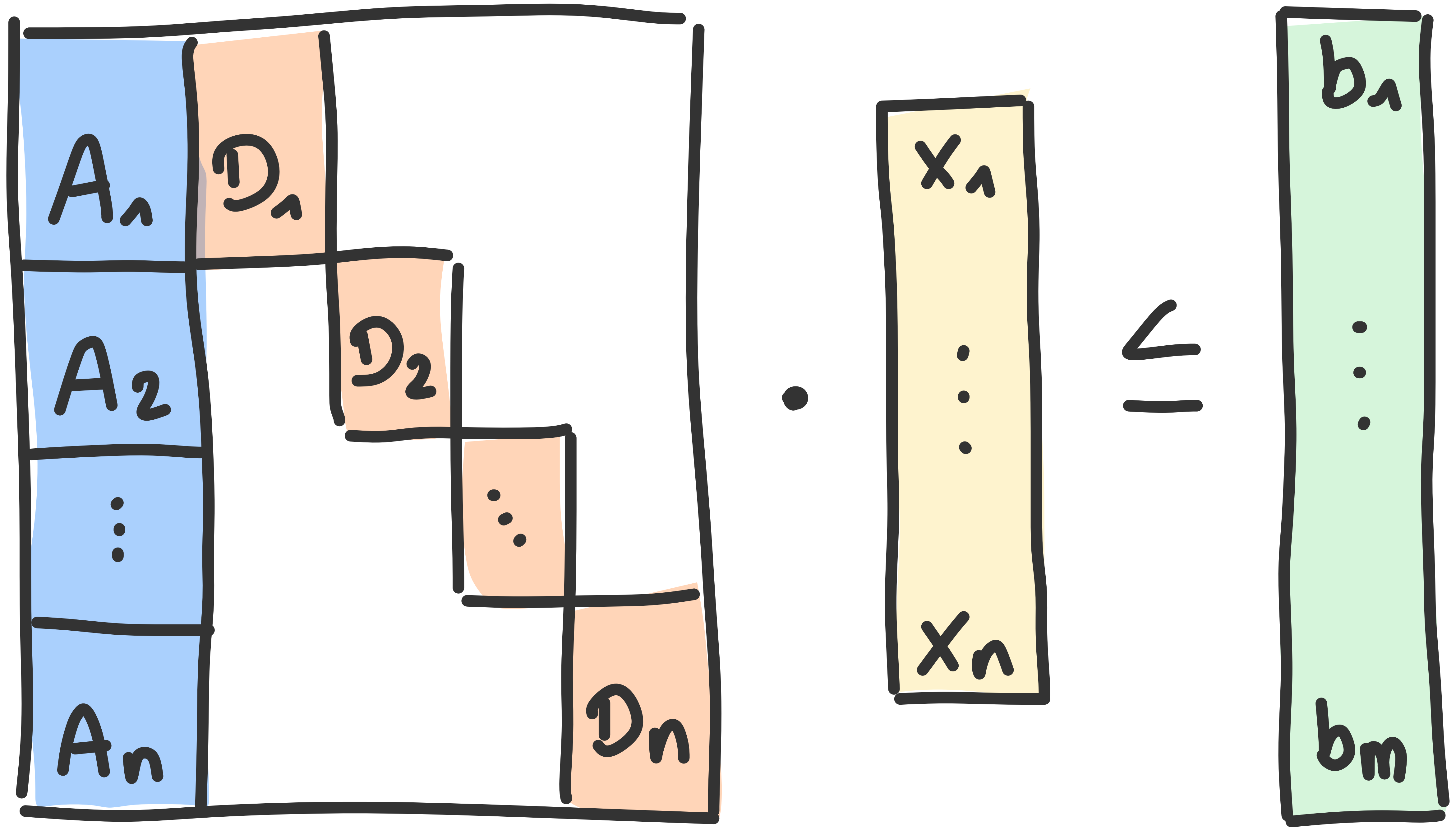

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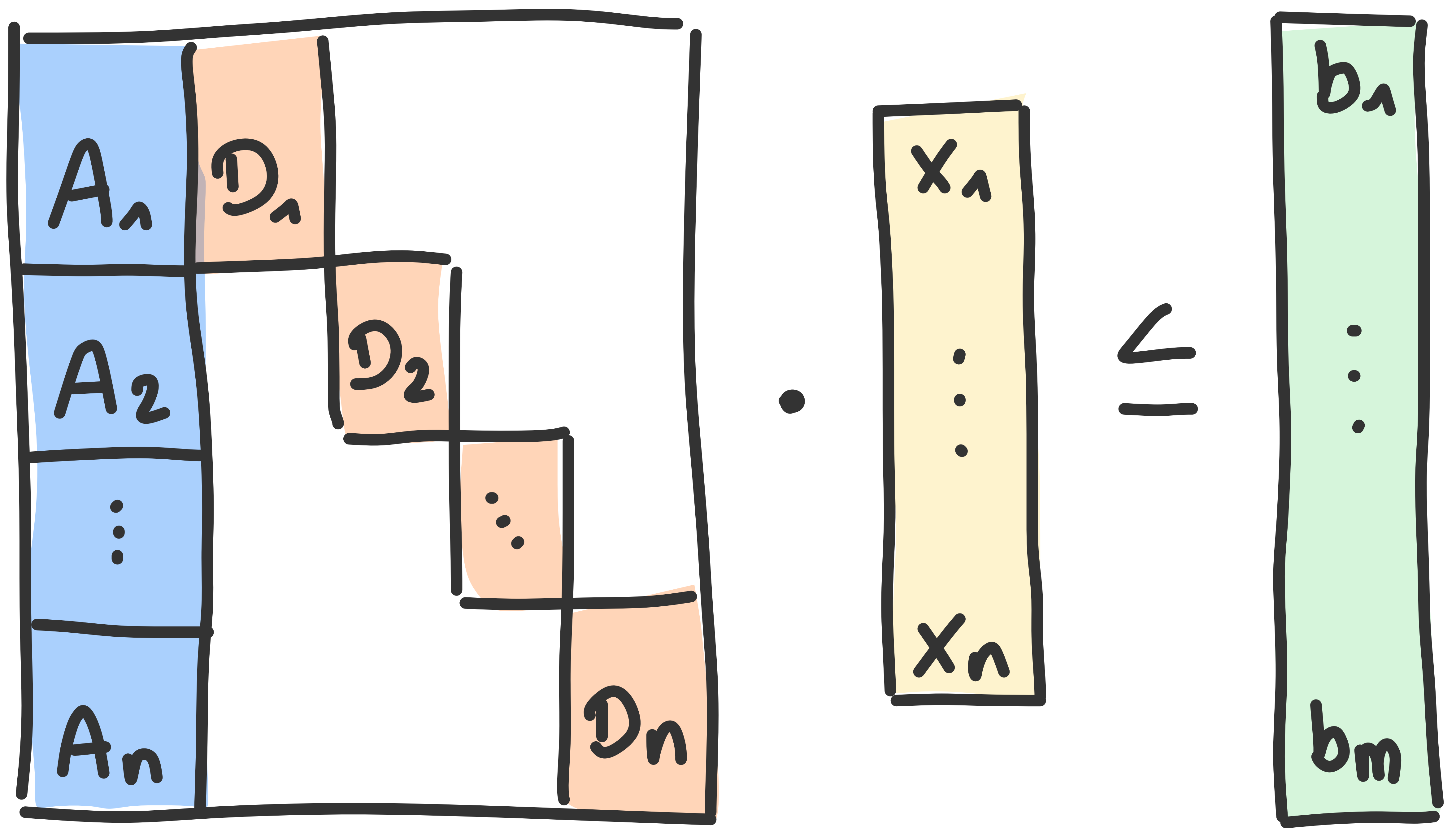
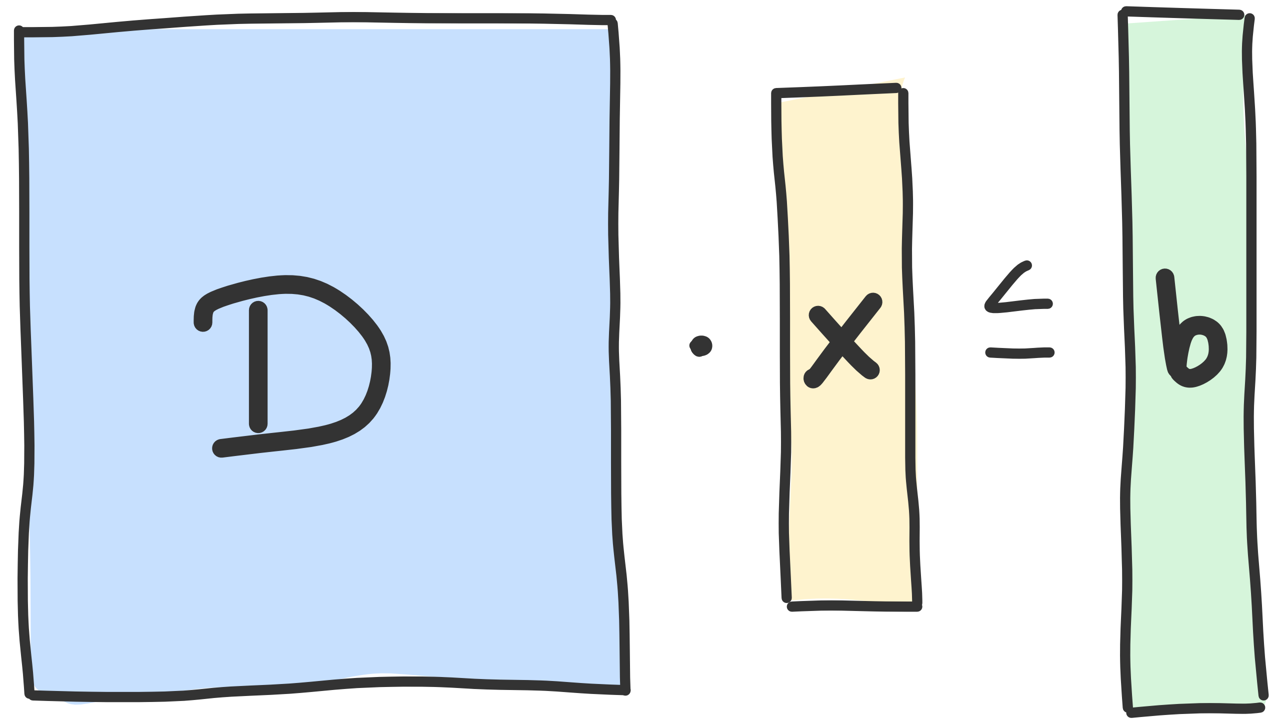


neural networks, worker scheduling, project planning, routing, facility location planning, …

The Integer Hull



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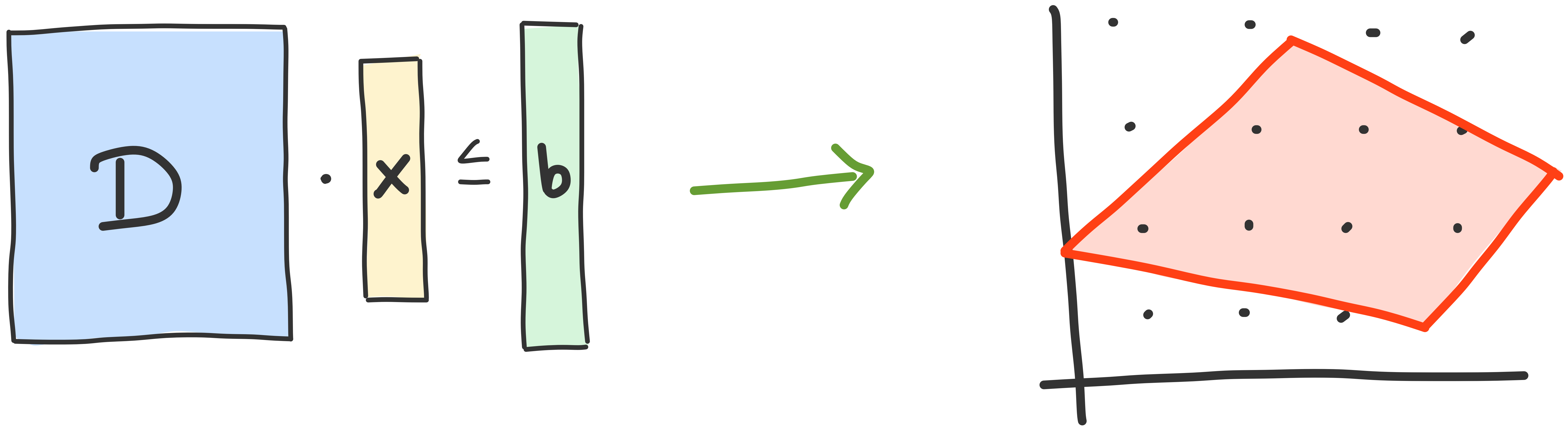

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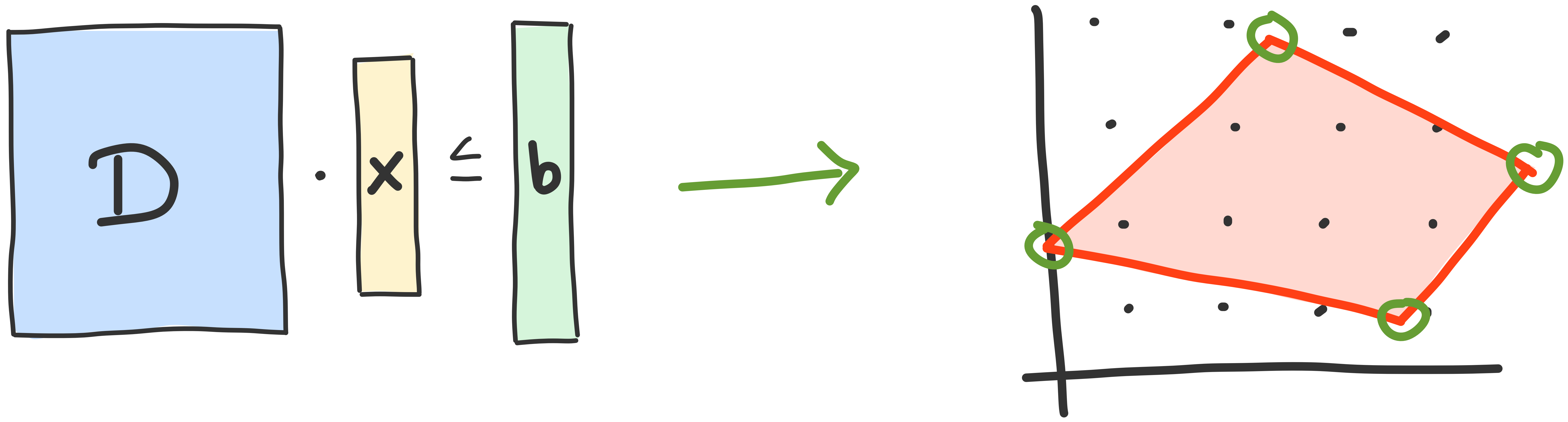

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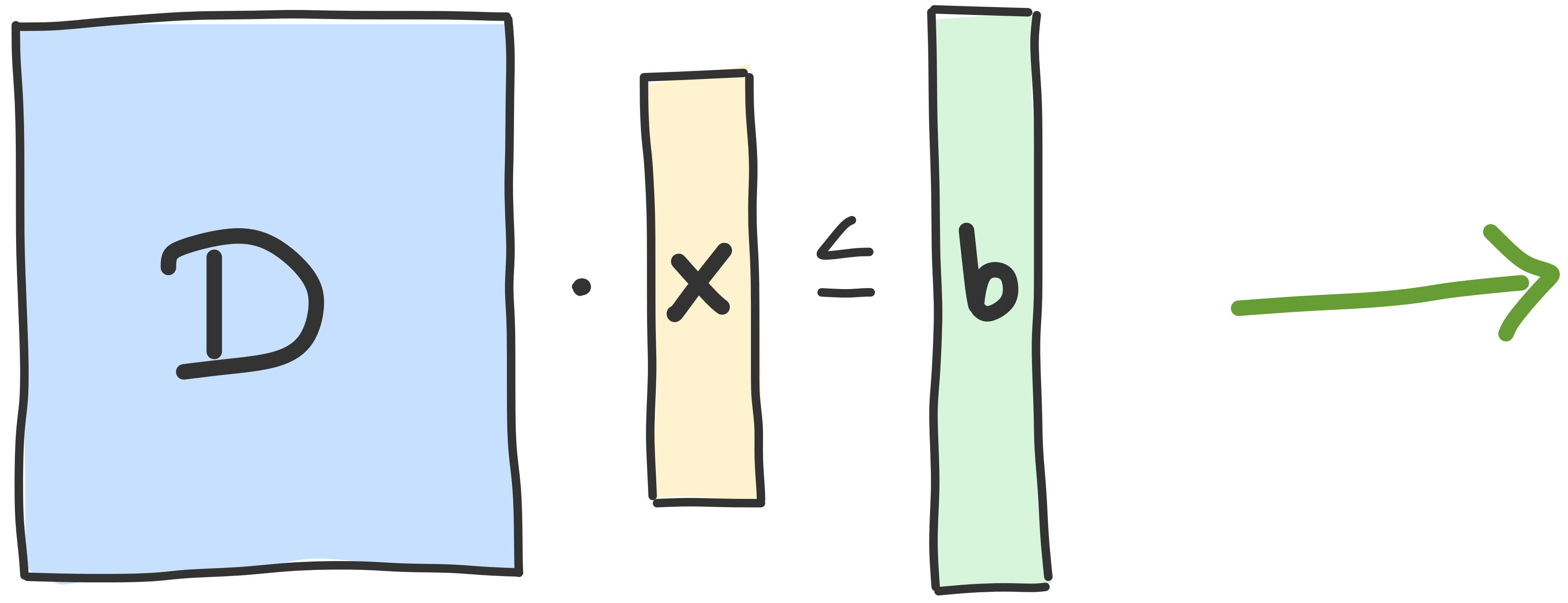

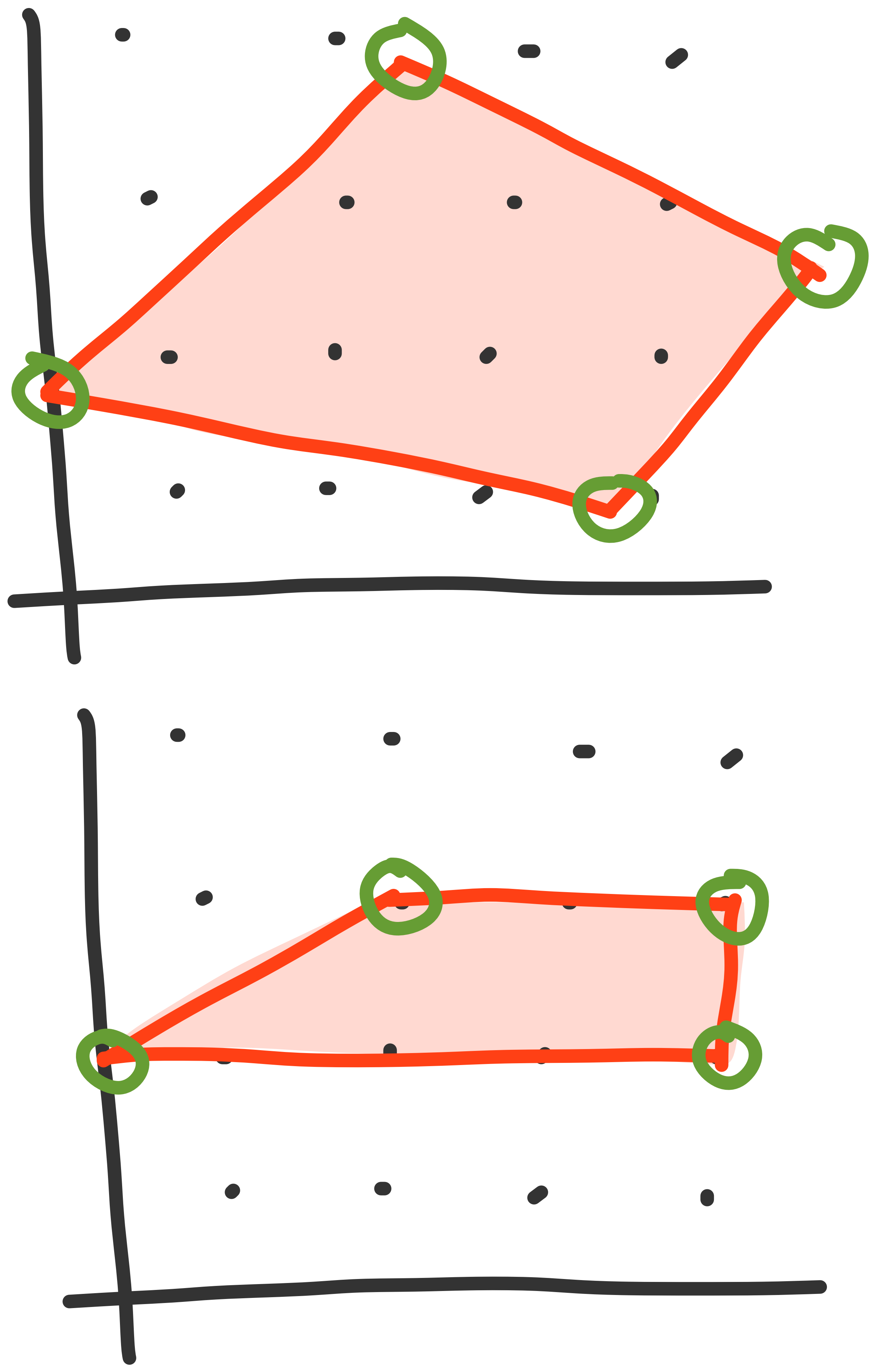

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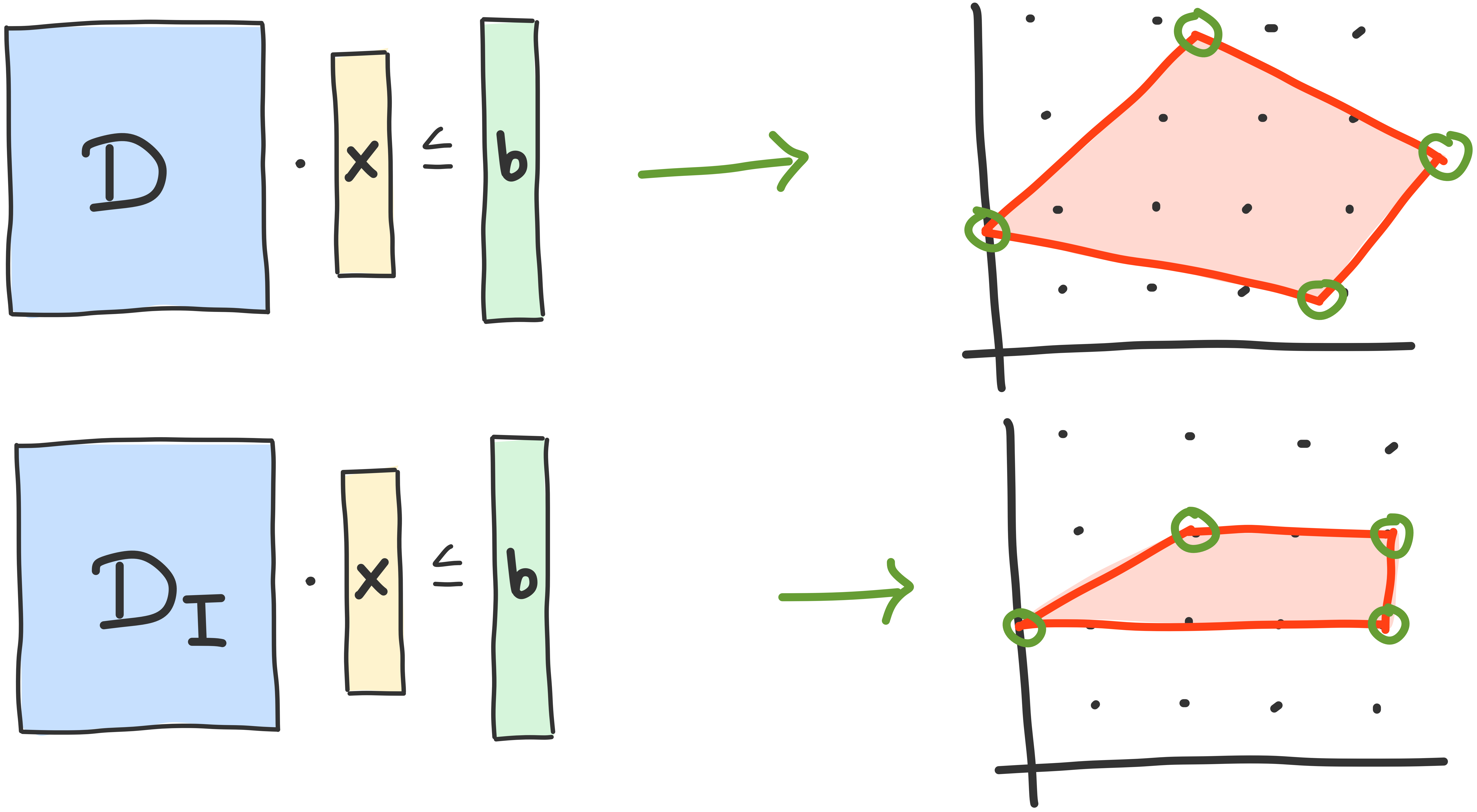

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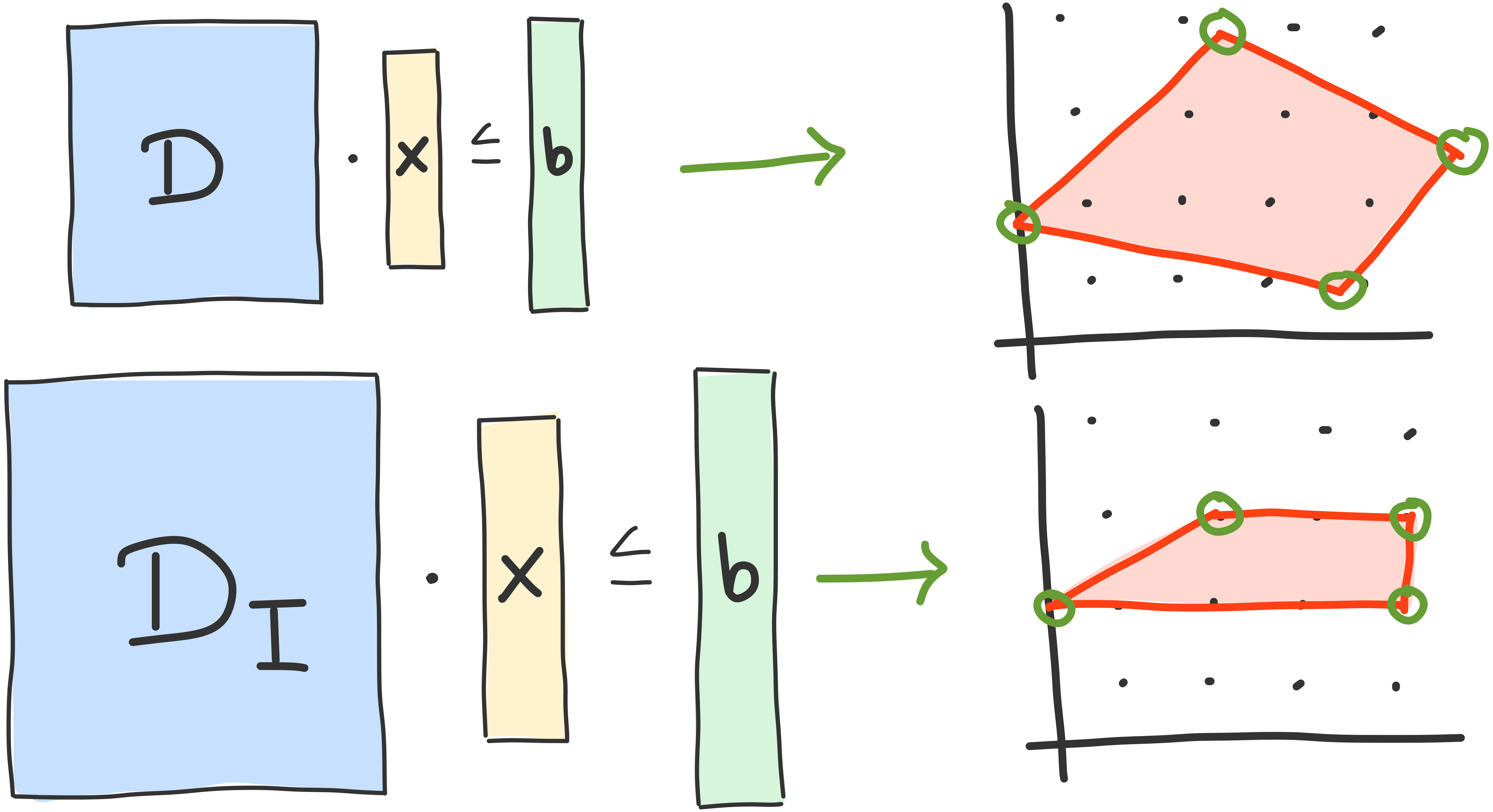

2-stage stochastic IPs

Theorem 1.* *Let $D \in \mathbb{Z}^{m\times k}$ with non-repeating rows and $\|A\|_\infty \leq \Delta$. There exists a $R \in \mathbb{N}$ depending on k and Δ such that, for each $r \in \{0,\dots,R-1\}^m$, there exist $B_r \in \mathbb{Z}^{m'\times k}$, $C_r \in \mathbb{Z}^{m'\times m}$ and $f_r \in \mathbb{Z}^{m'}$ such that the following holds:*

For each $b \in \mathbb{Z}^m$ with $b - r \in R \cdot \mathbb{Z}^m$, one has

$$\underbrace{P(b)_I}_{\text{integer hull}} = \{x \in \mathbb{R}^k \colon B_r x \leq f_r + C_r b\}\,.$$

* "A parameterized linear formulation of the integer hull" by F. Eisenbrand and T. Rothvoss, 2025

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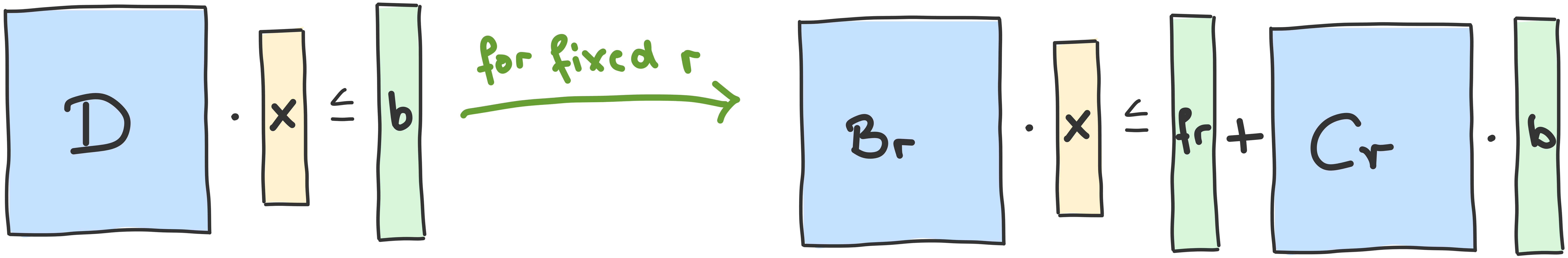

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There is an fpt-sized description of the integer hull of $Dx \leq b$ (for fixed r)

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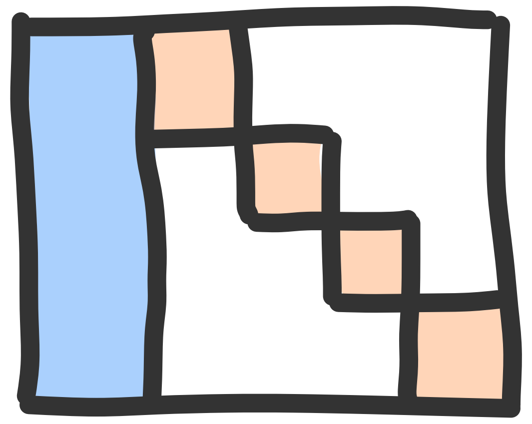
$$P(b)_I = \{x \in \mathbb{R}^k \colon B_r x \leq f_r + C_r b\} .$$

Furthermore, the number $R \in \mathbb{N}$, *the matrices* B_r *and* C_r, *as well as the vector* f_r *can be computed in time depending on* Δ *and* k *only.*

"A parameterized linear formulation of the integer hull" by F. Eisenbrand and T. Rothvoss, 2025

The Algorithm



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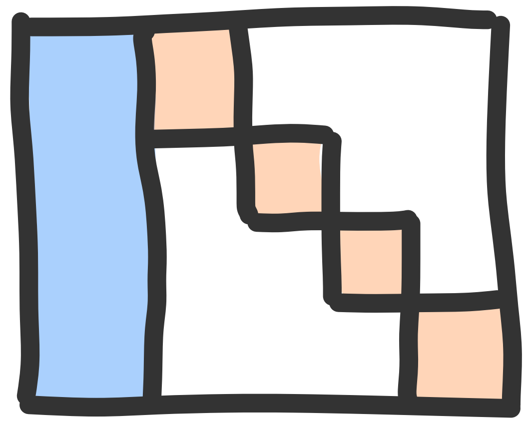

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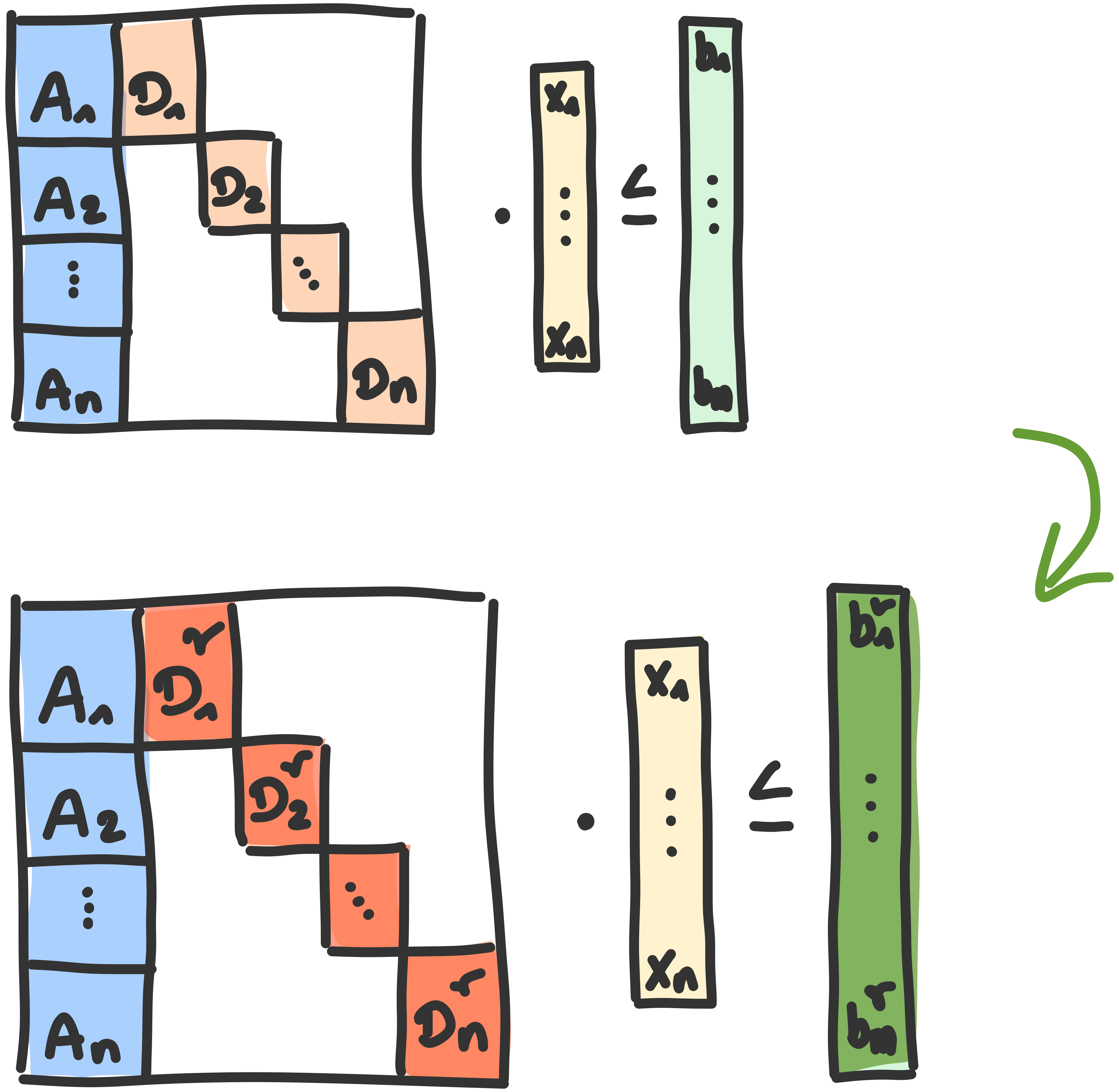

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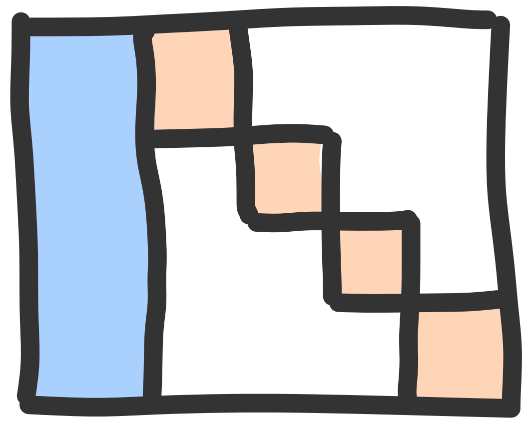

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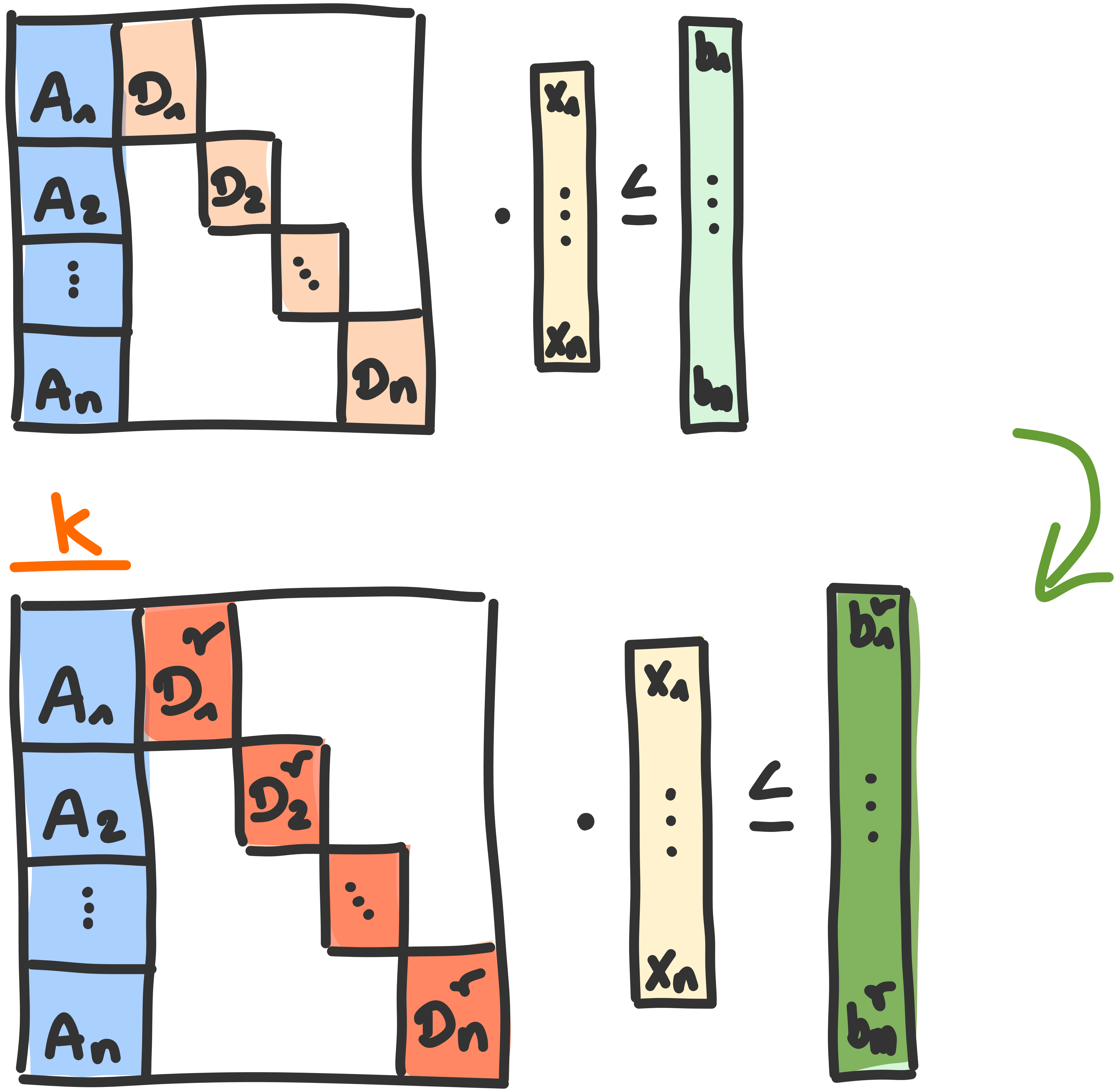
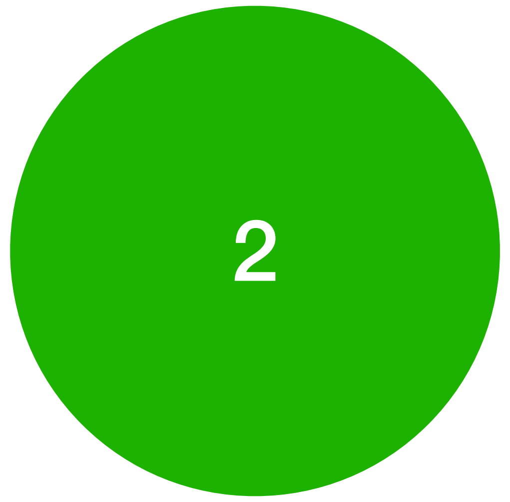
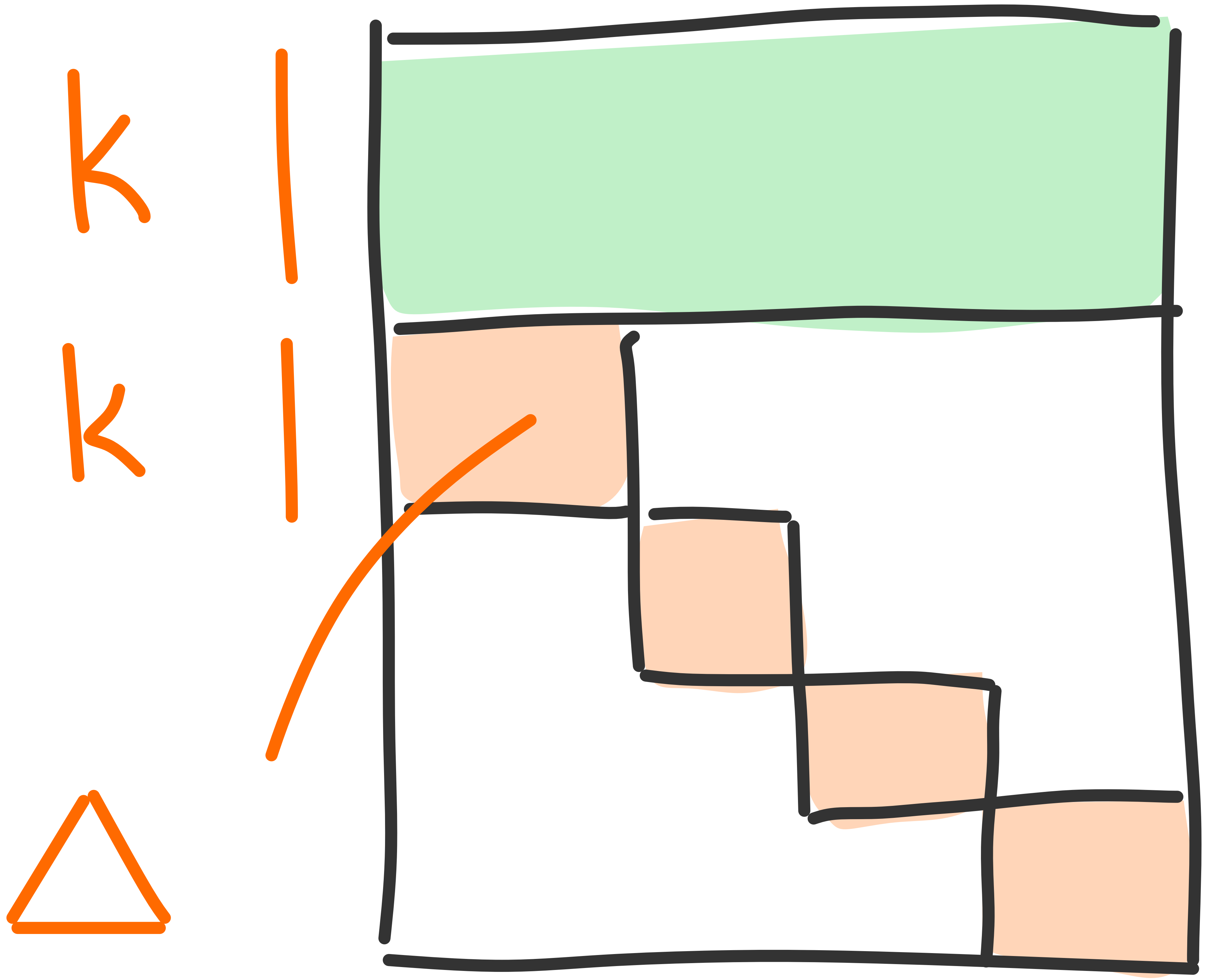
2. Compute integer hull description of each D_i and replace them

3. Solve the MIP







n-fold IPs

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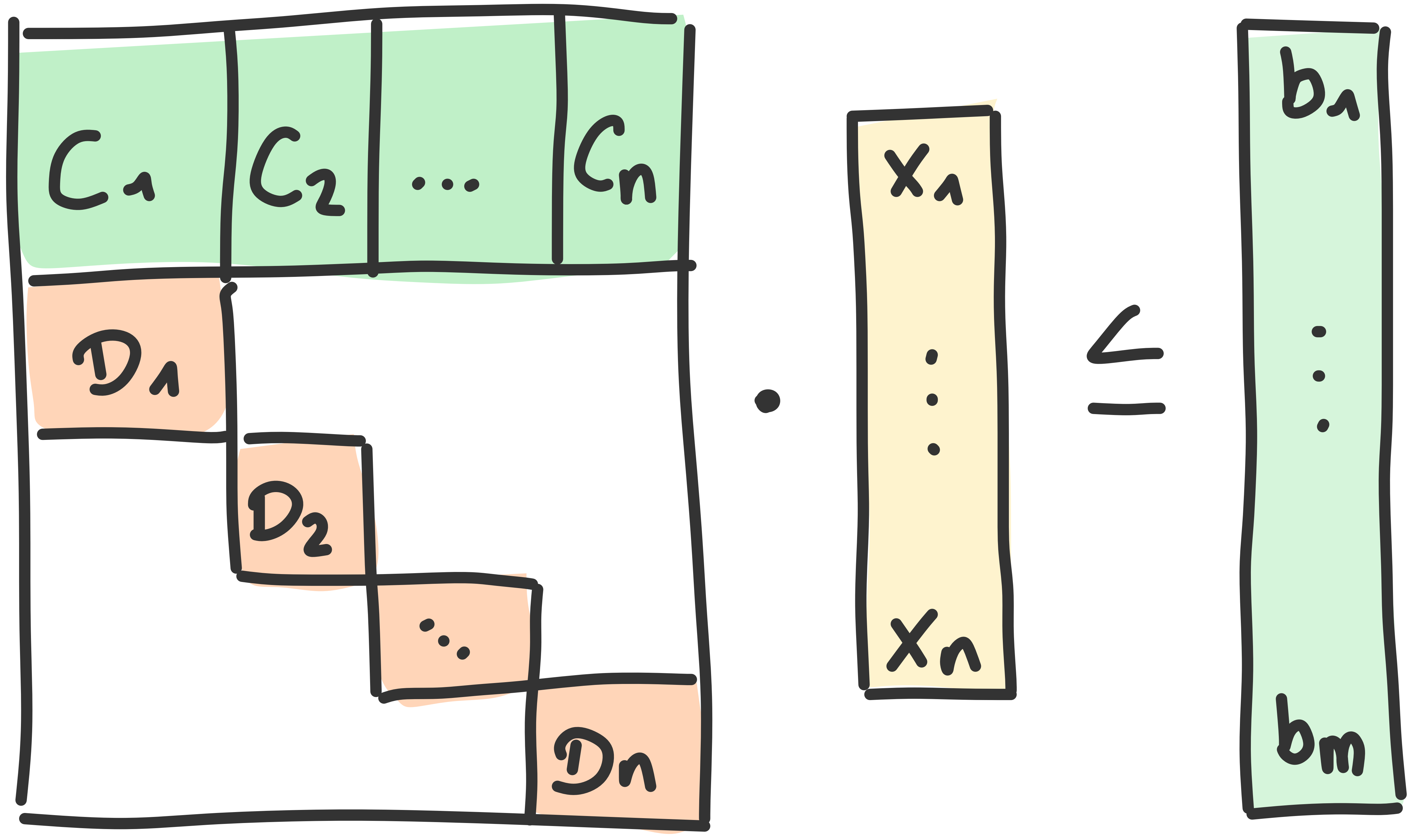

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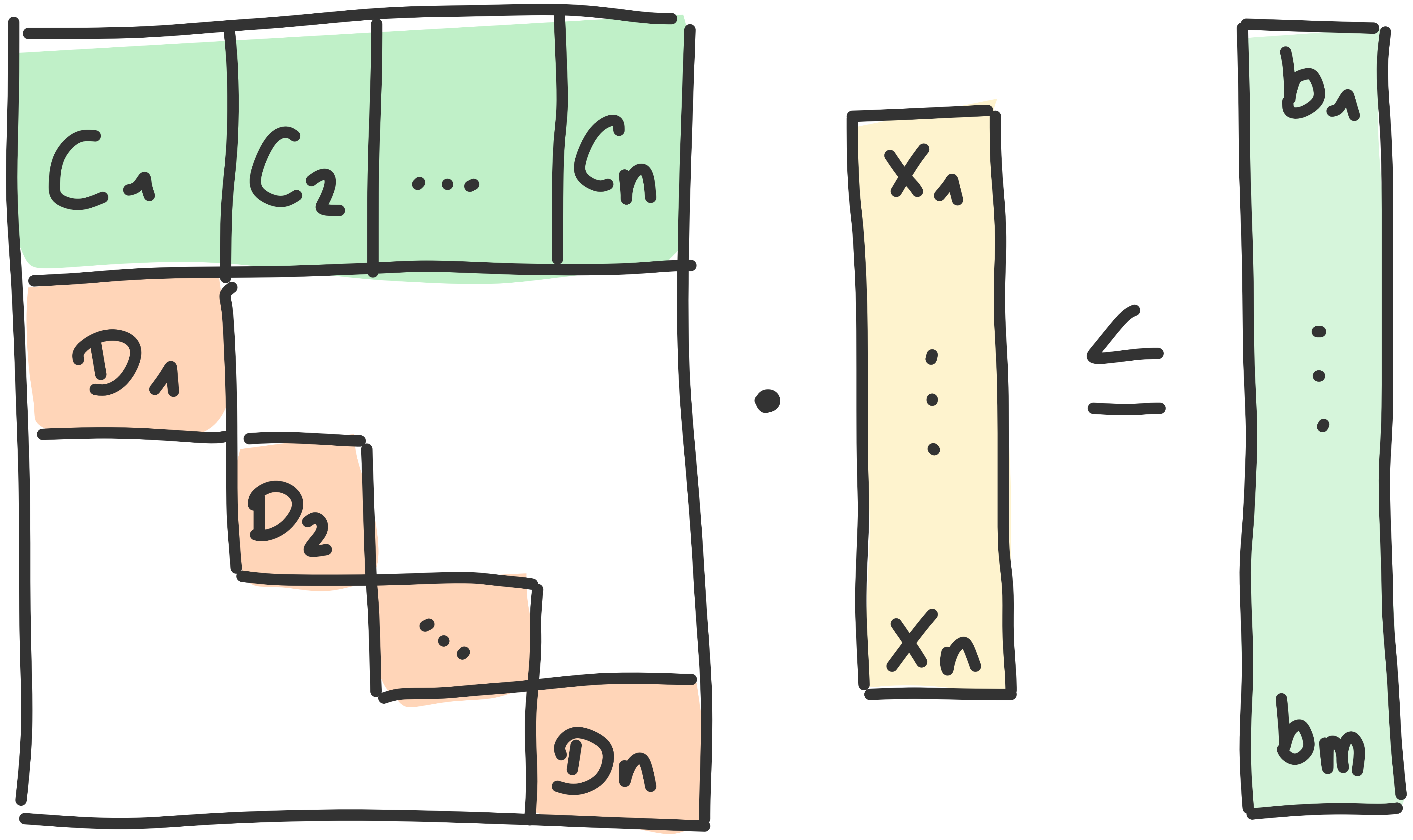

scheduling, knapsack-like problems, string problems, social choice, …

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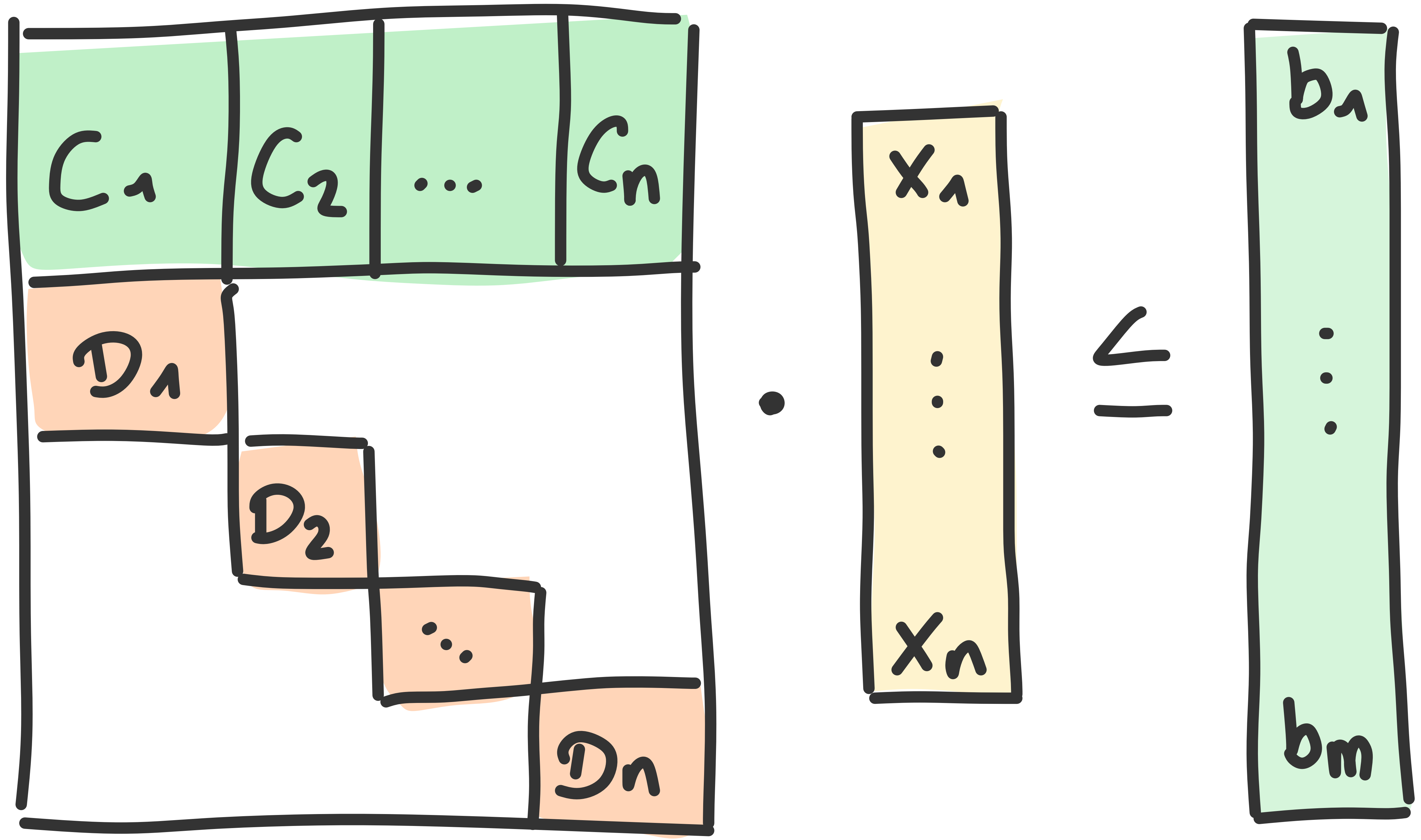

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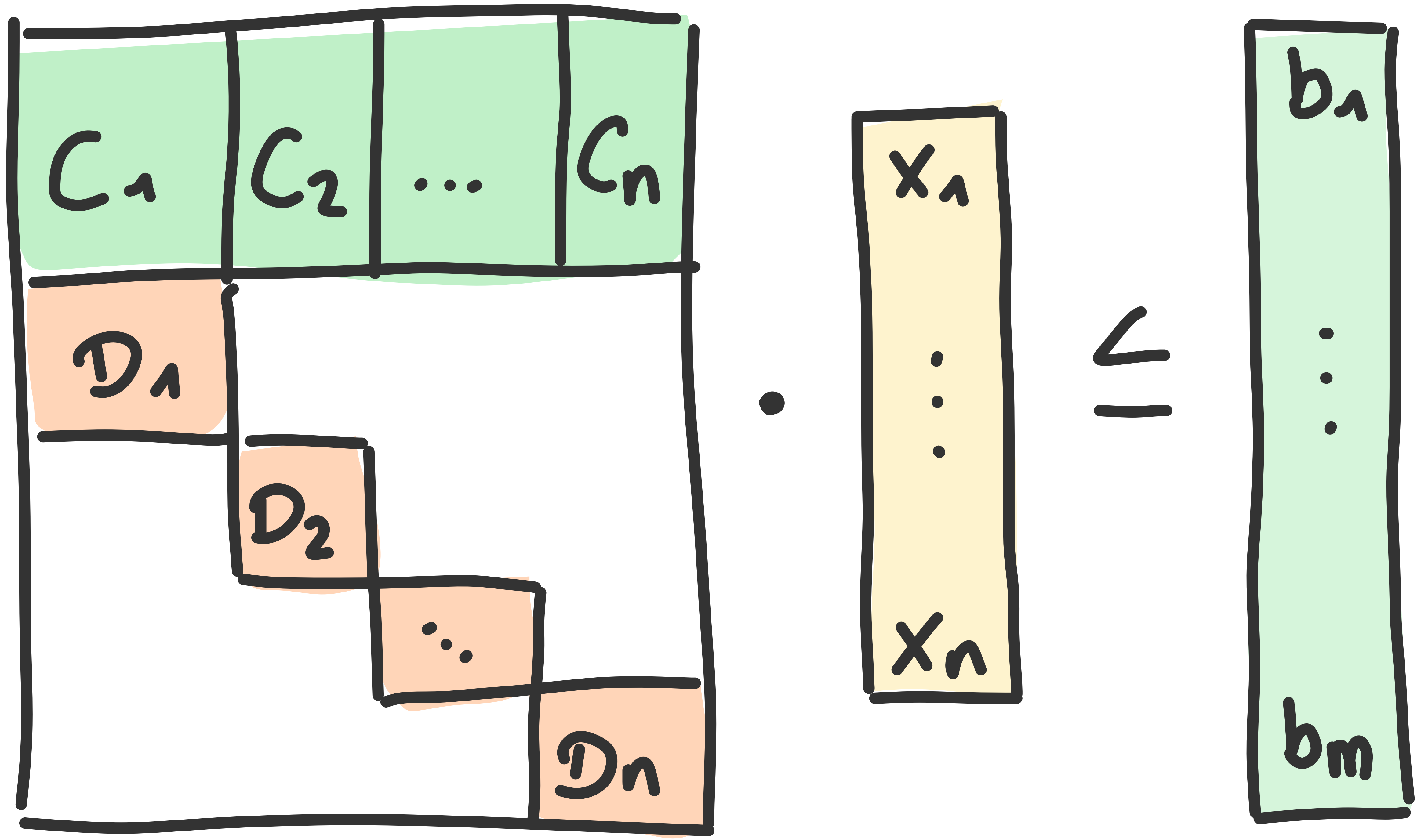

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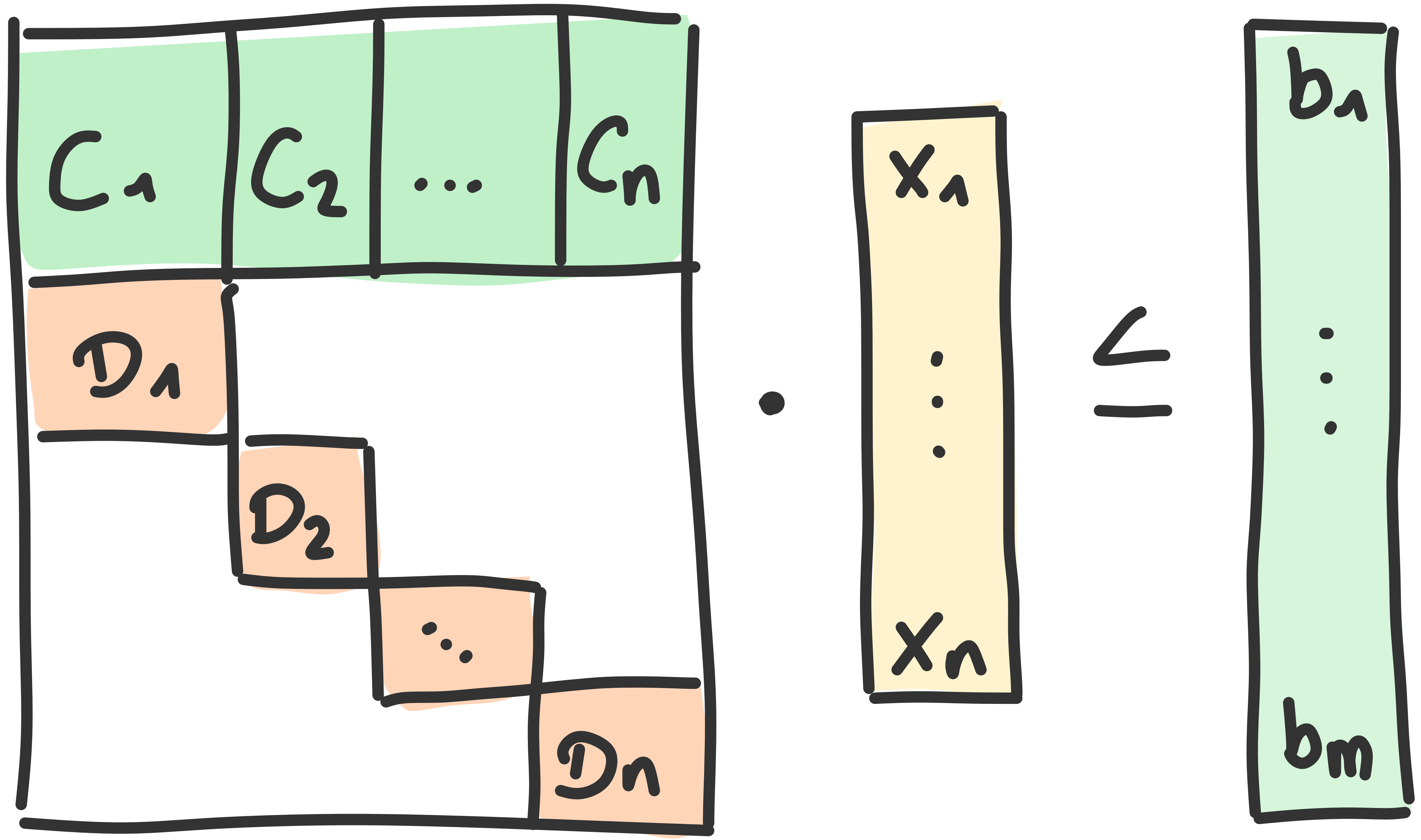

If dimension is small, we are done

If it is large, there are only few types of diagonal blocks

n-fold IPs

$\exists b', b''$ **sign-compatible to** b **s.t.** $b = b' + b''$ **and** $\forall v \in \mathbb{Z}^y_{\geq 0}$ **with**

$D_i v = b$ **there exist** v', v'' **with 1.** $v = v' + v''$ **2.** $D_i v' = b'$ **3.** $D_i v'' = b''$

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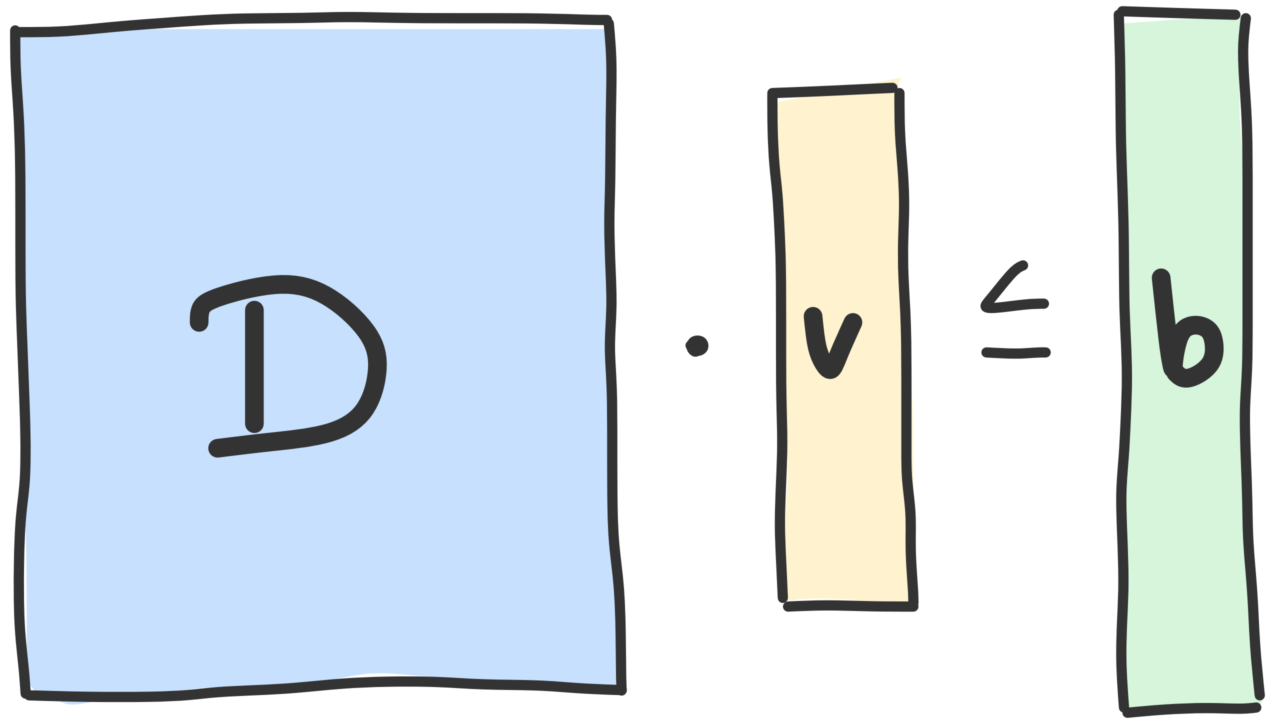

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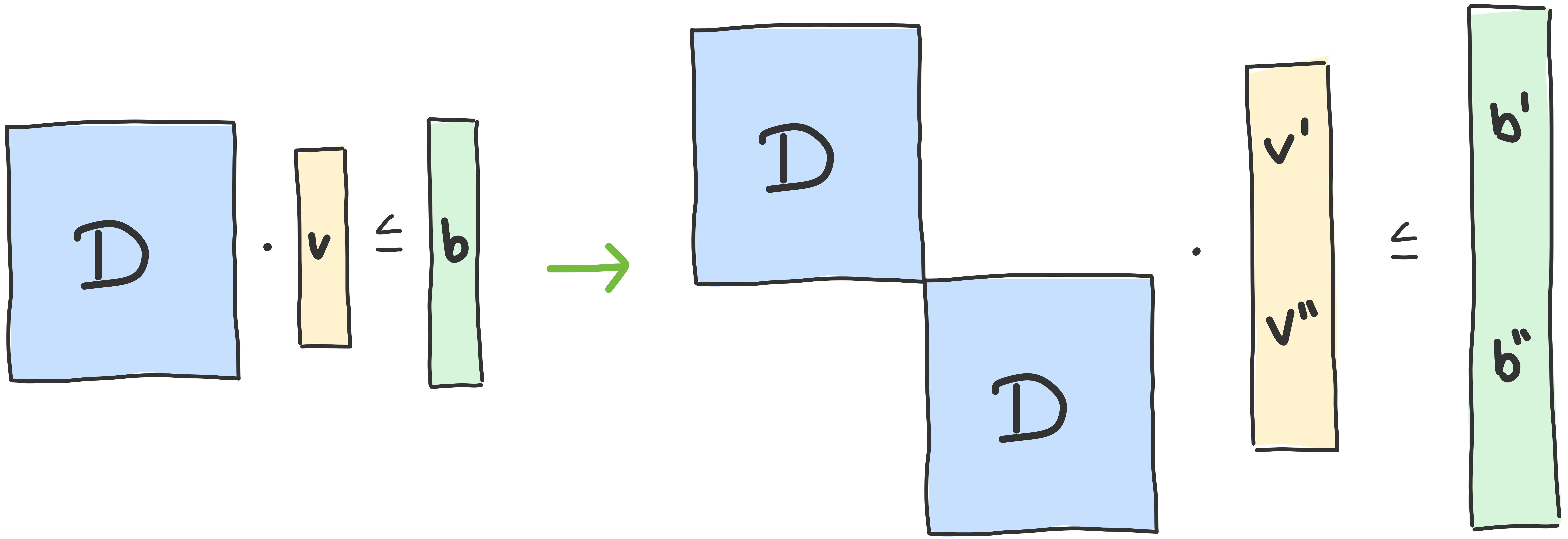

n-fold IPs - based on Klein Lemma

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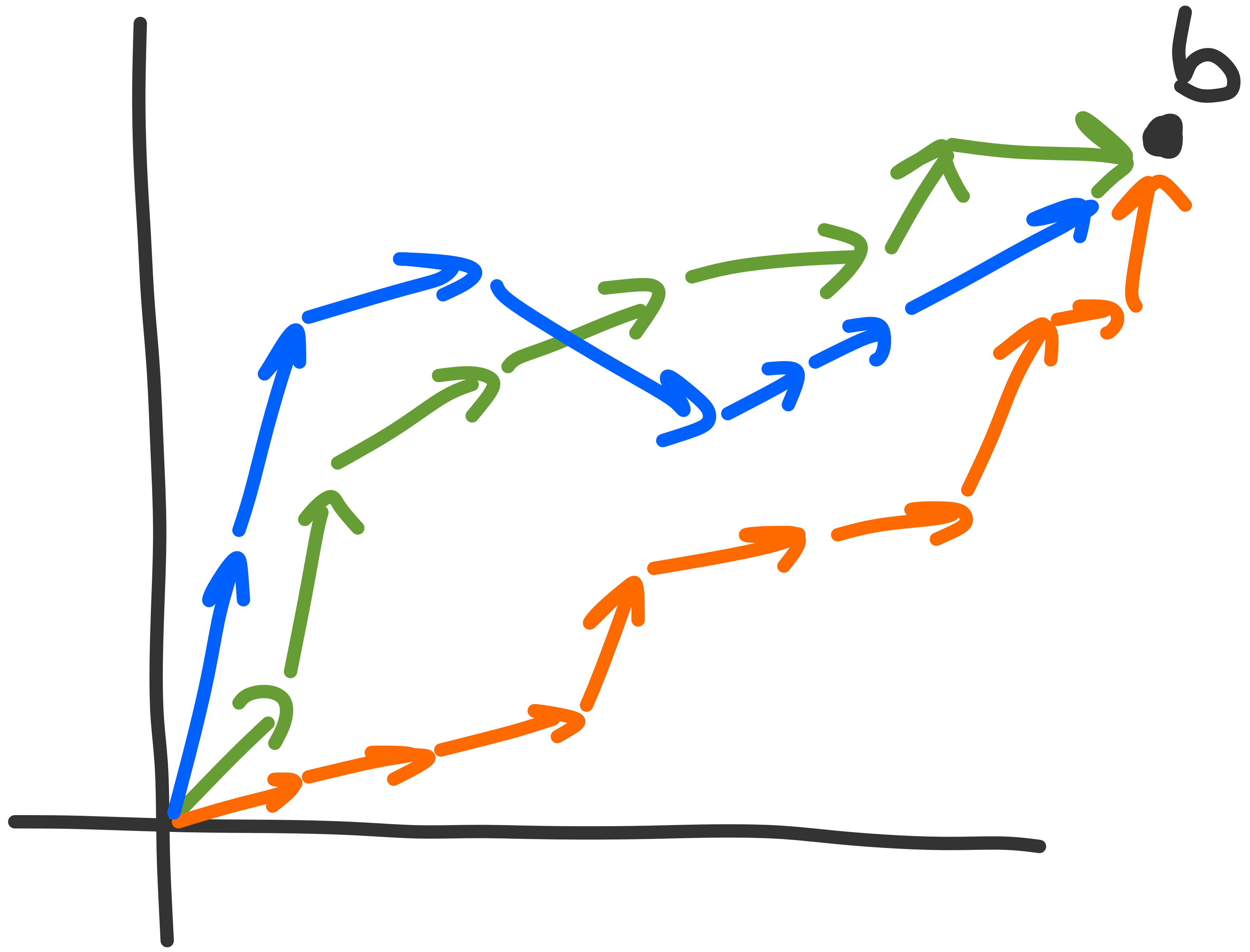

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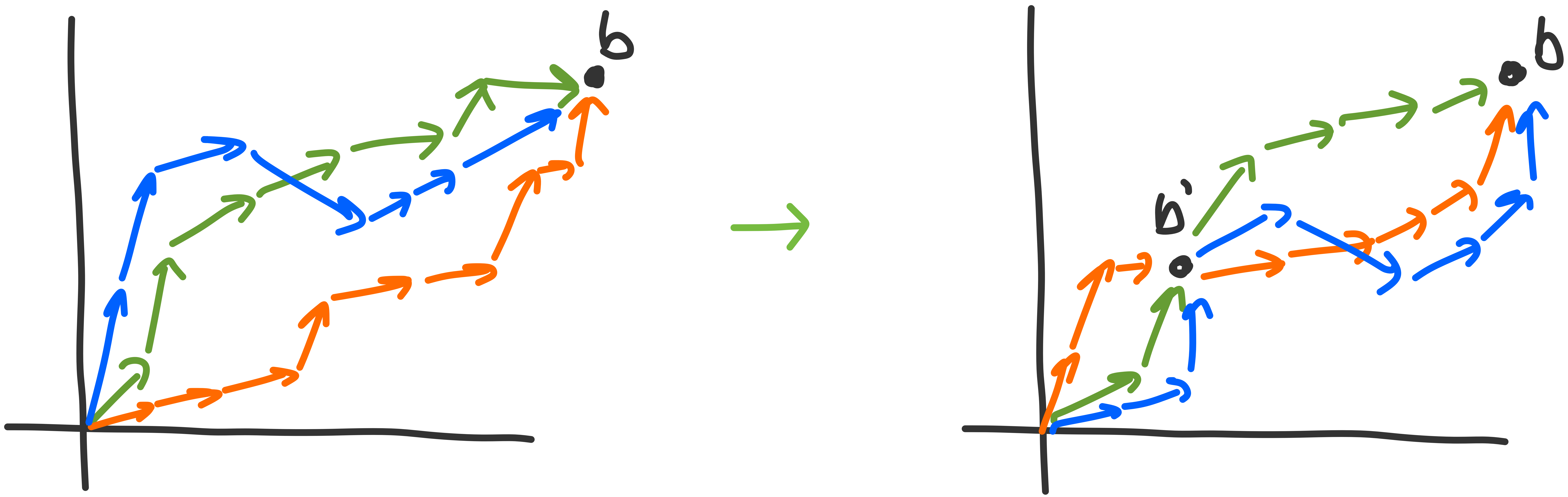

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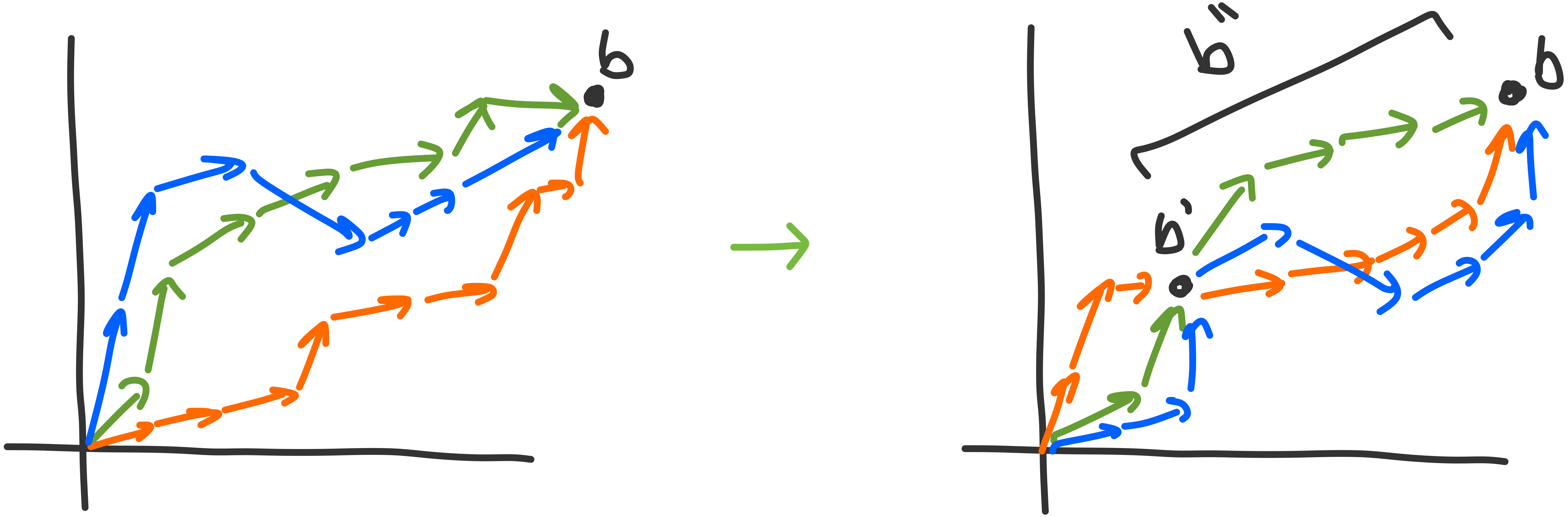
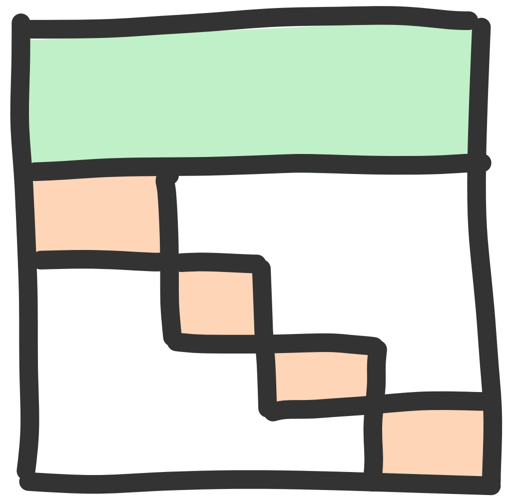
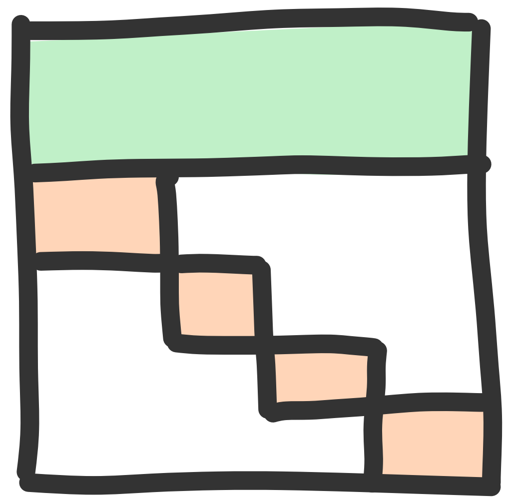


The Algorithm



1. Breaking the bricks

The Algorithm



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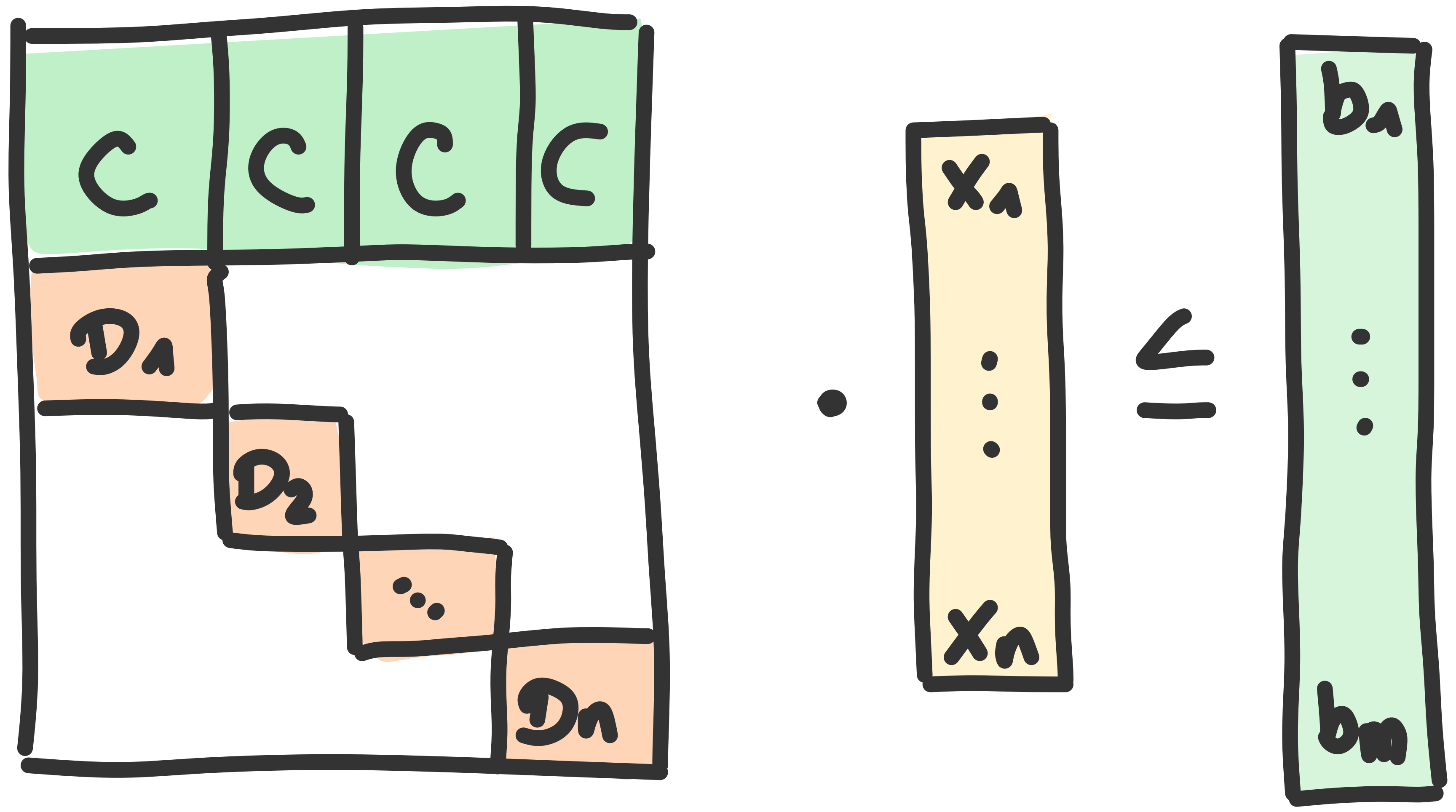

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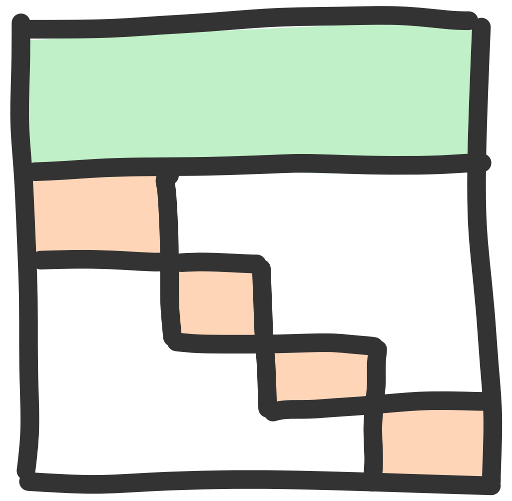

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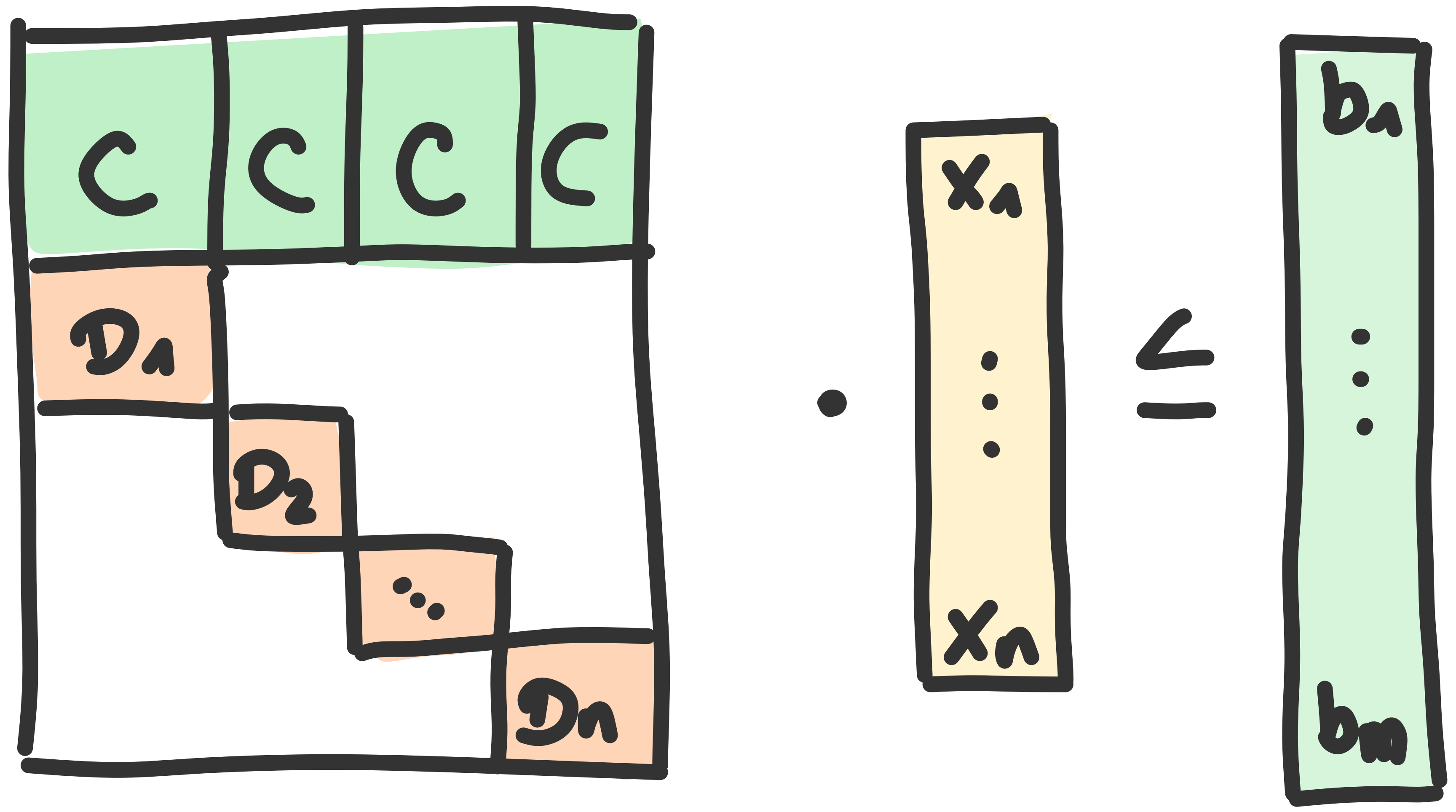
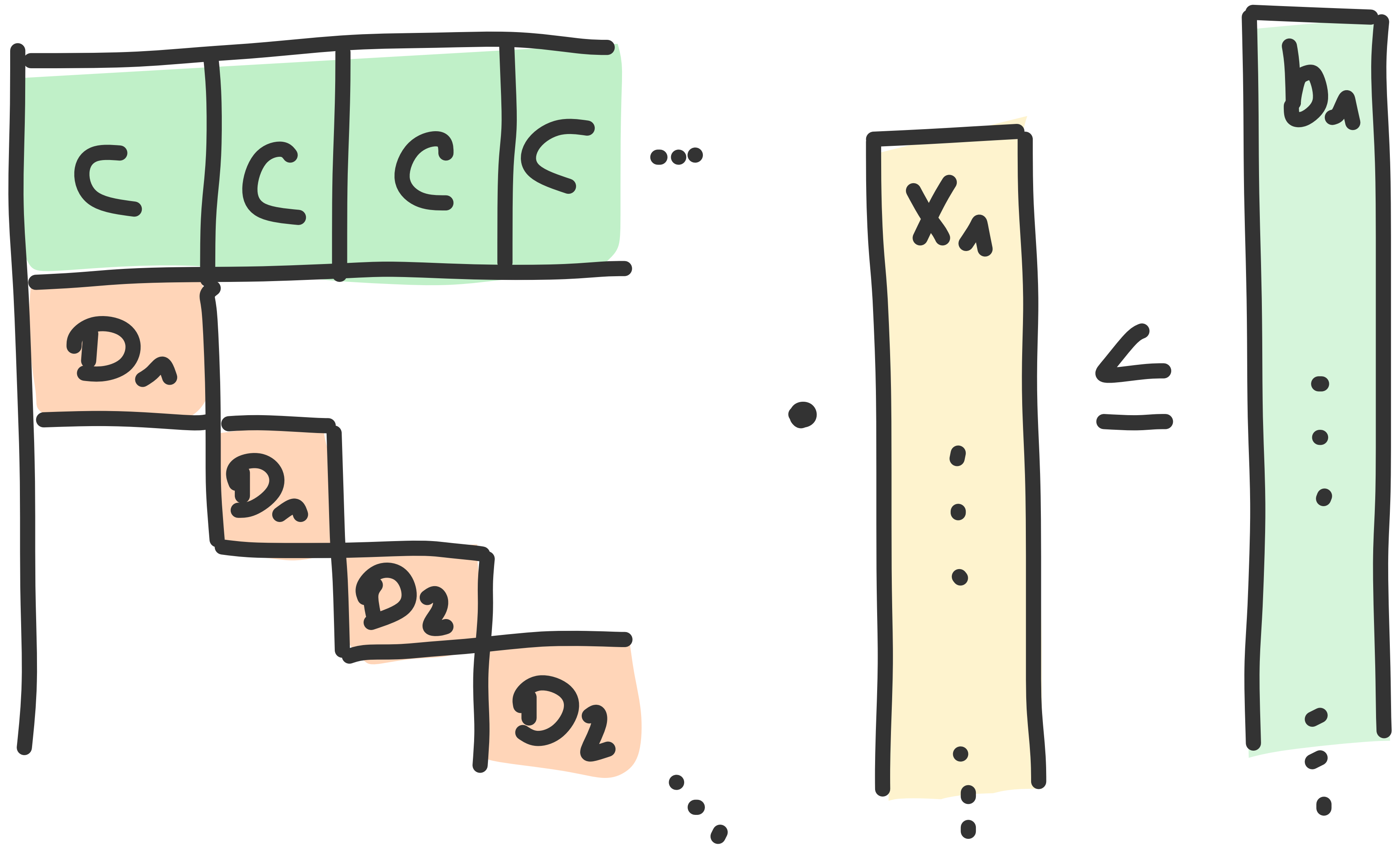
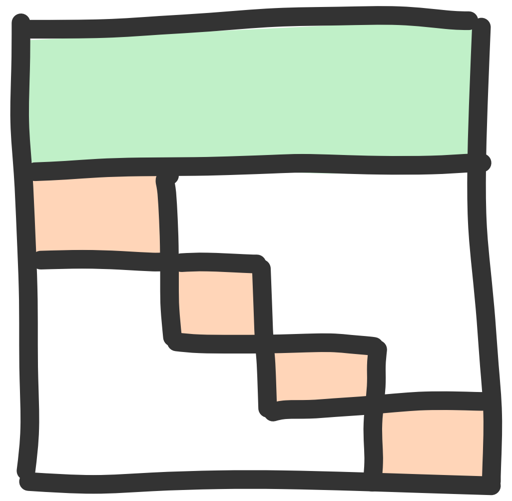
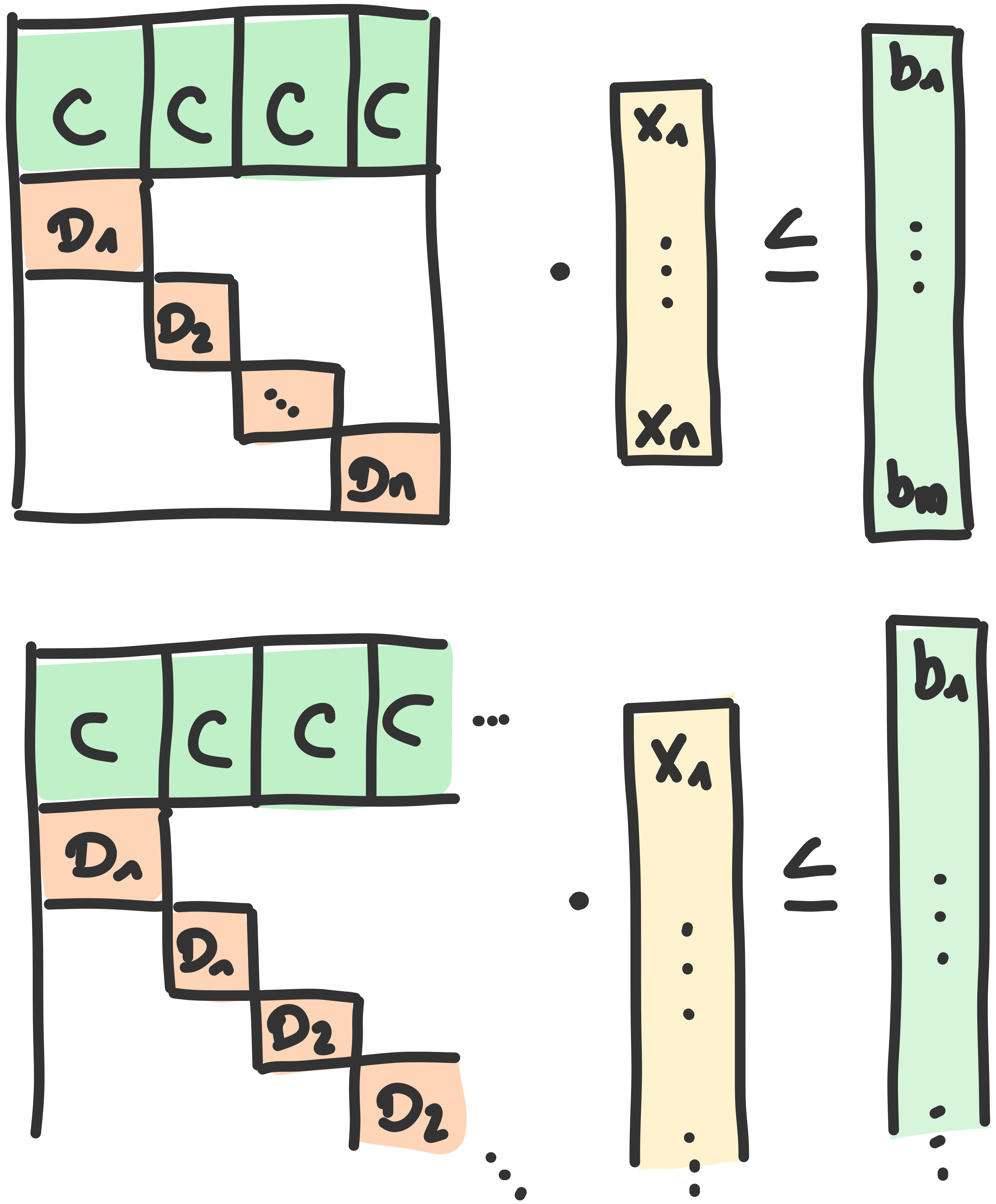




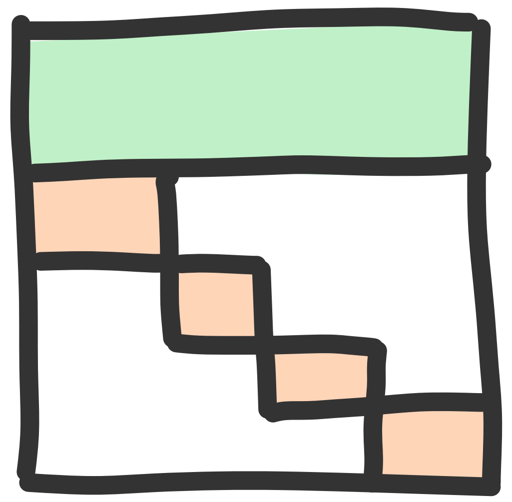


The Algorithm

1. Breaking the bricks

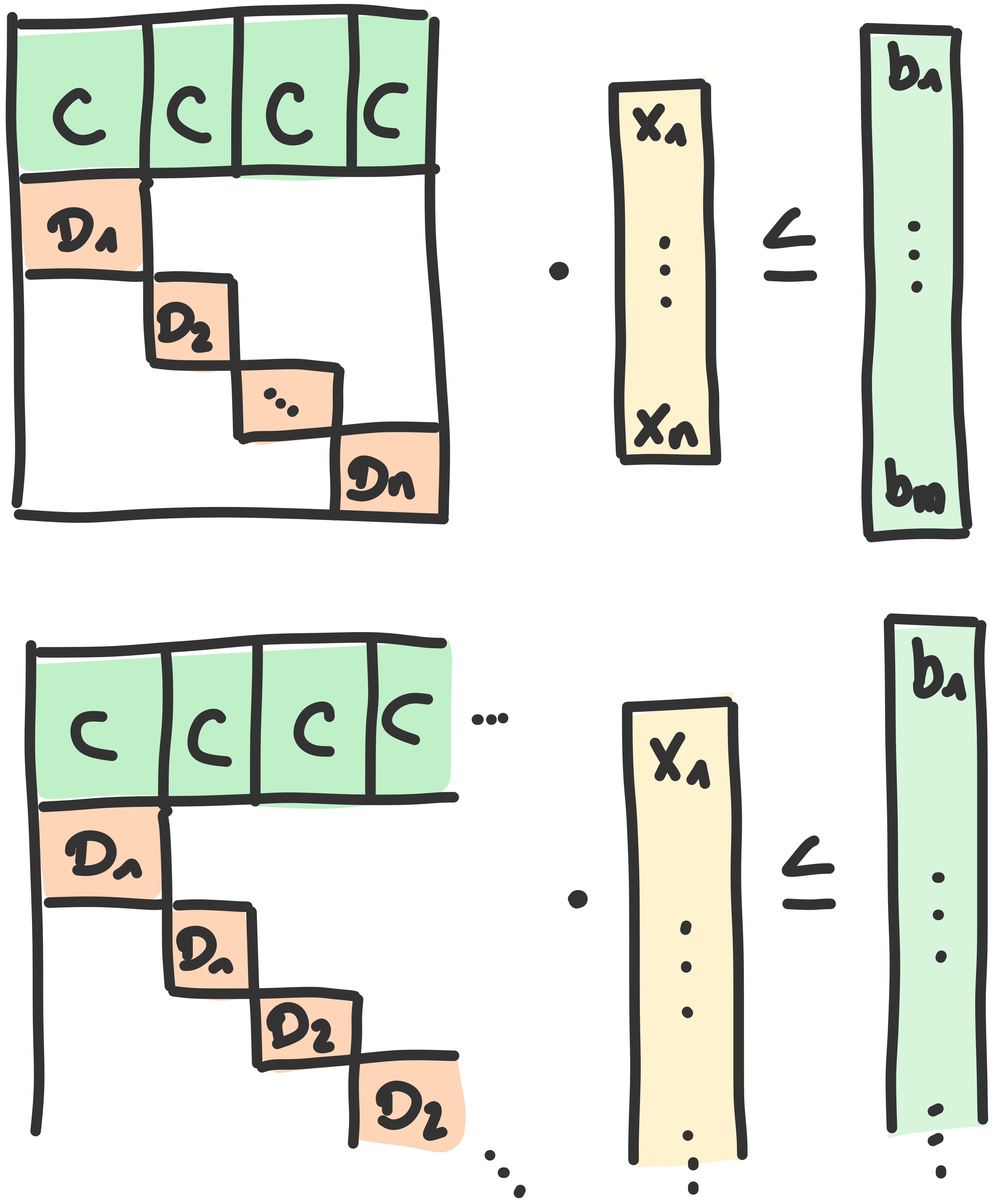


The Algorithm

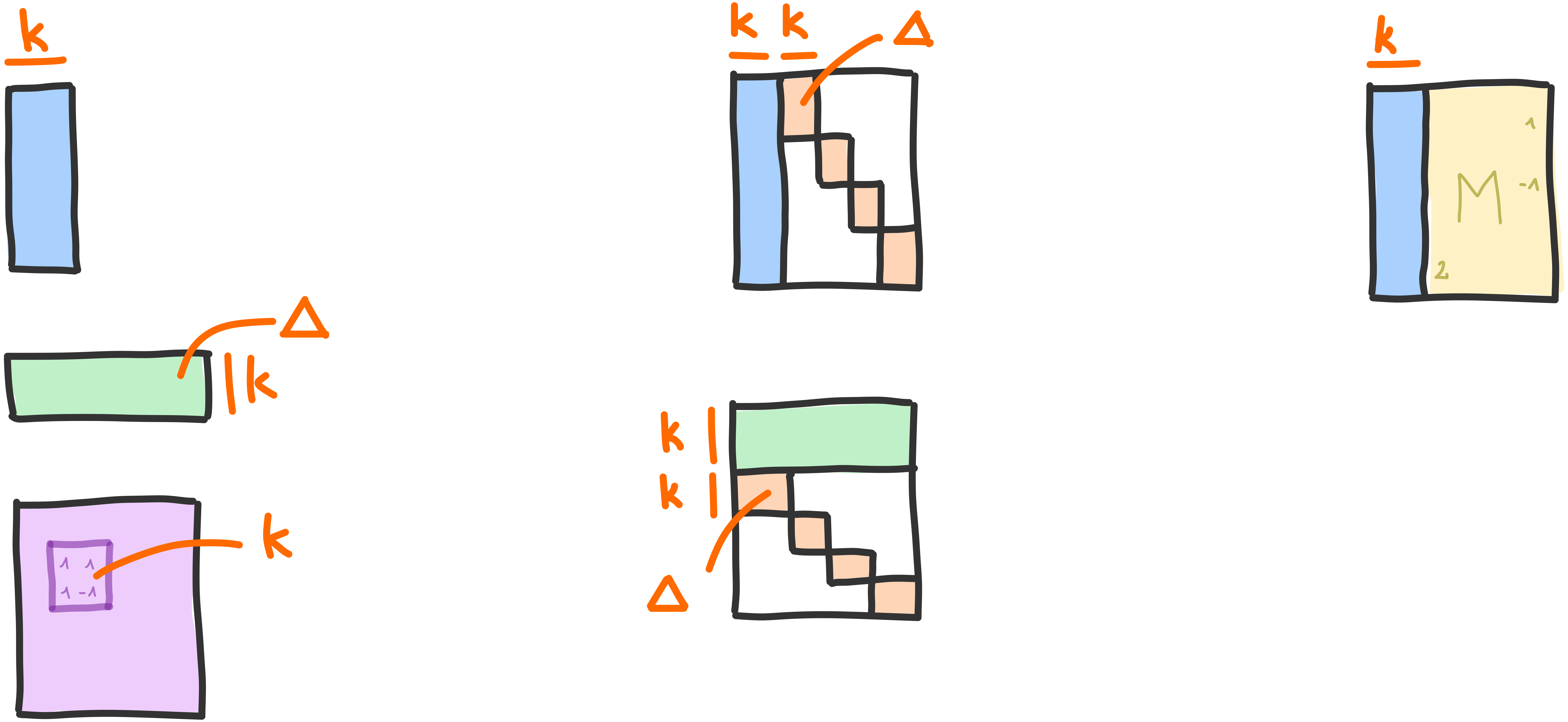


1. Breaking the bricks

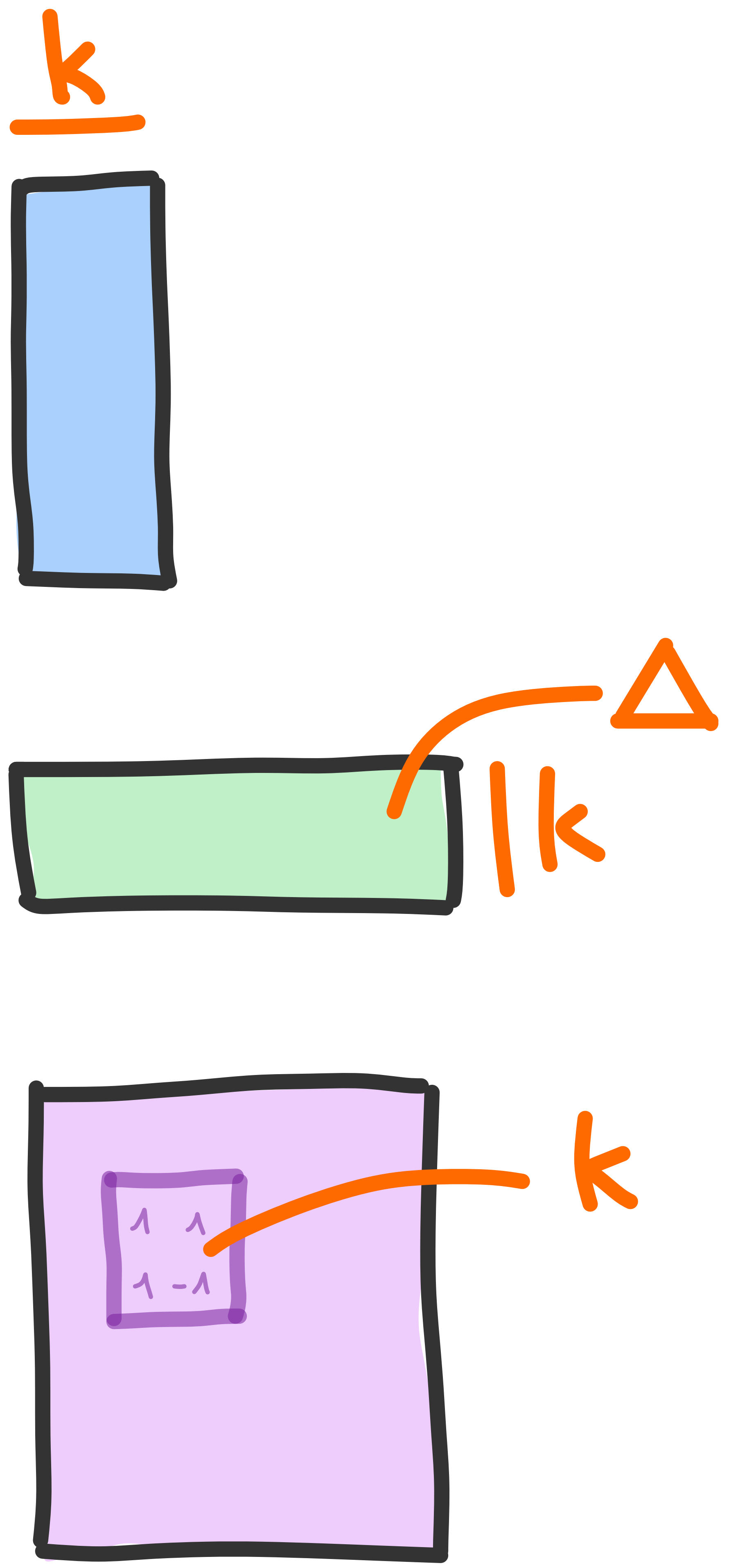
2. We can deal easily with the high-multiplicity instance :)

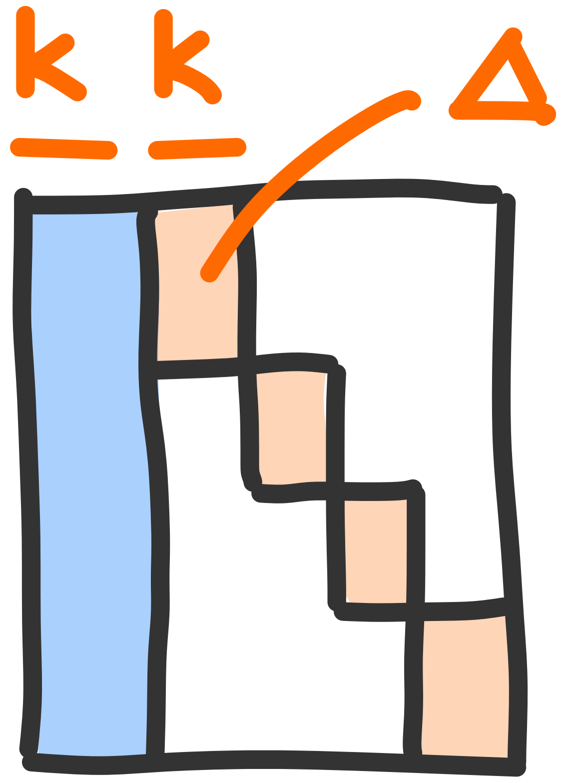


Integer Programming meets FPT

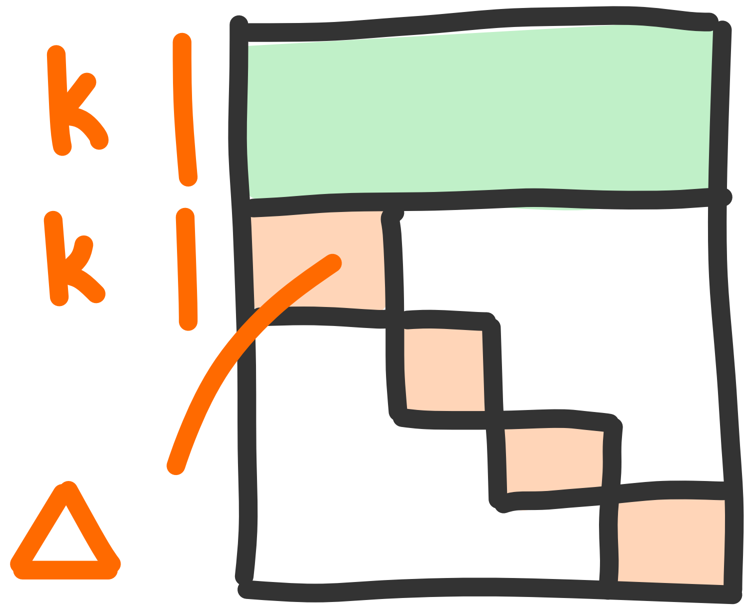


Integer Programming meets FPT

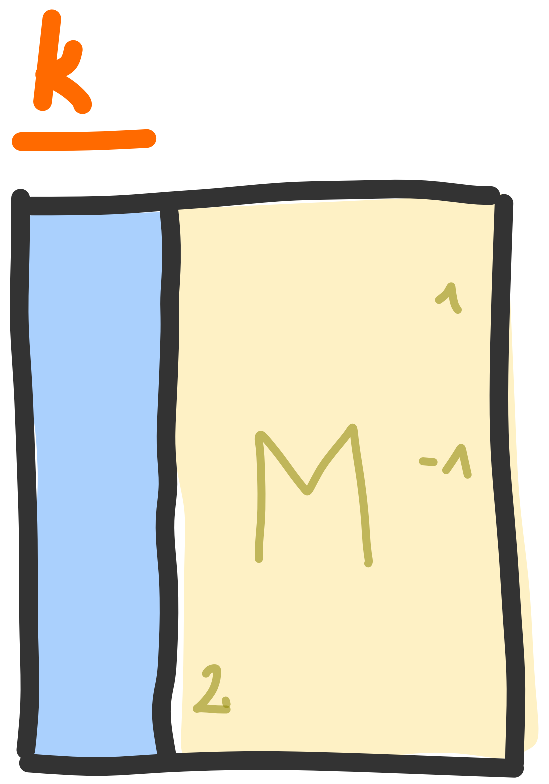




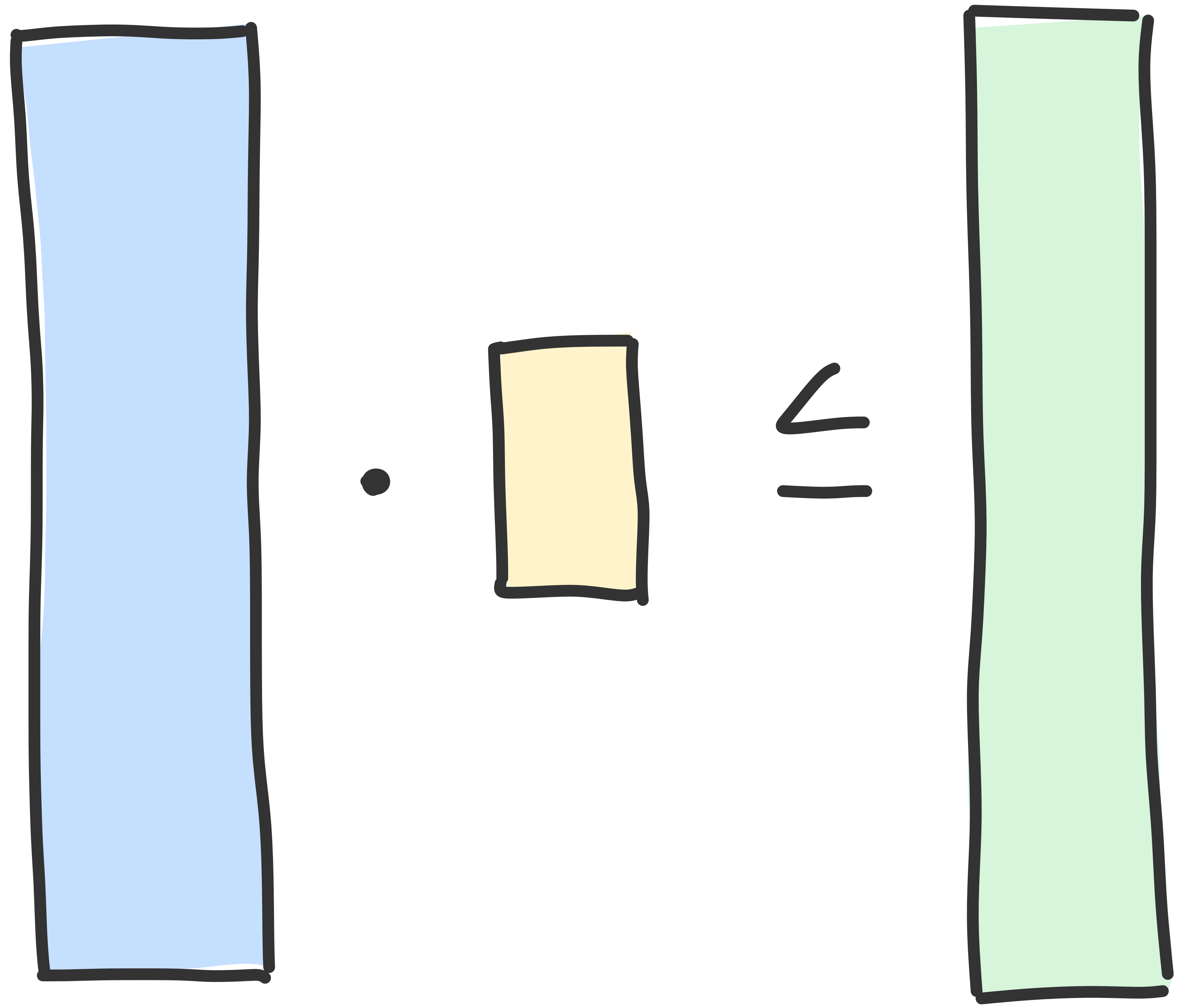
Integer
Hull



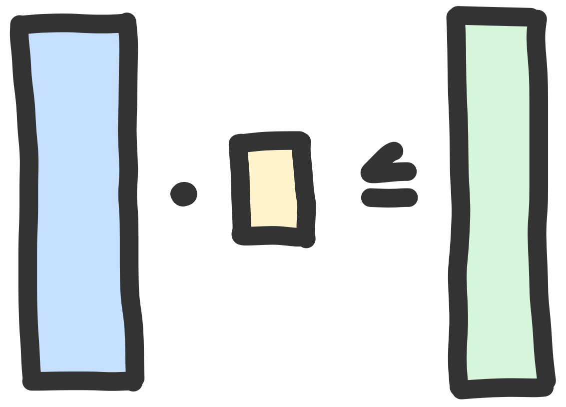
Brick
Decomposition



Lenstra-type Integer Programs



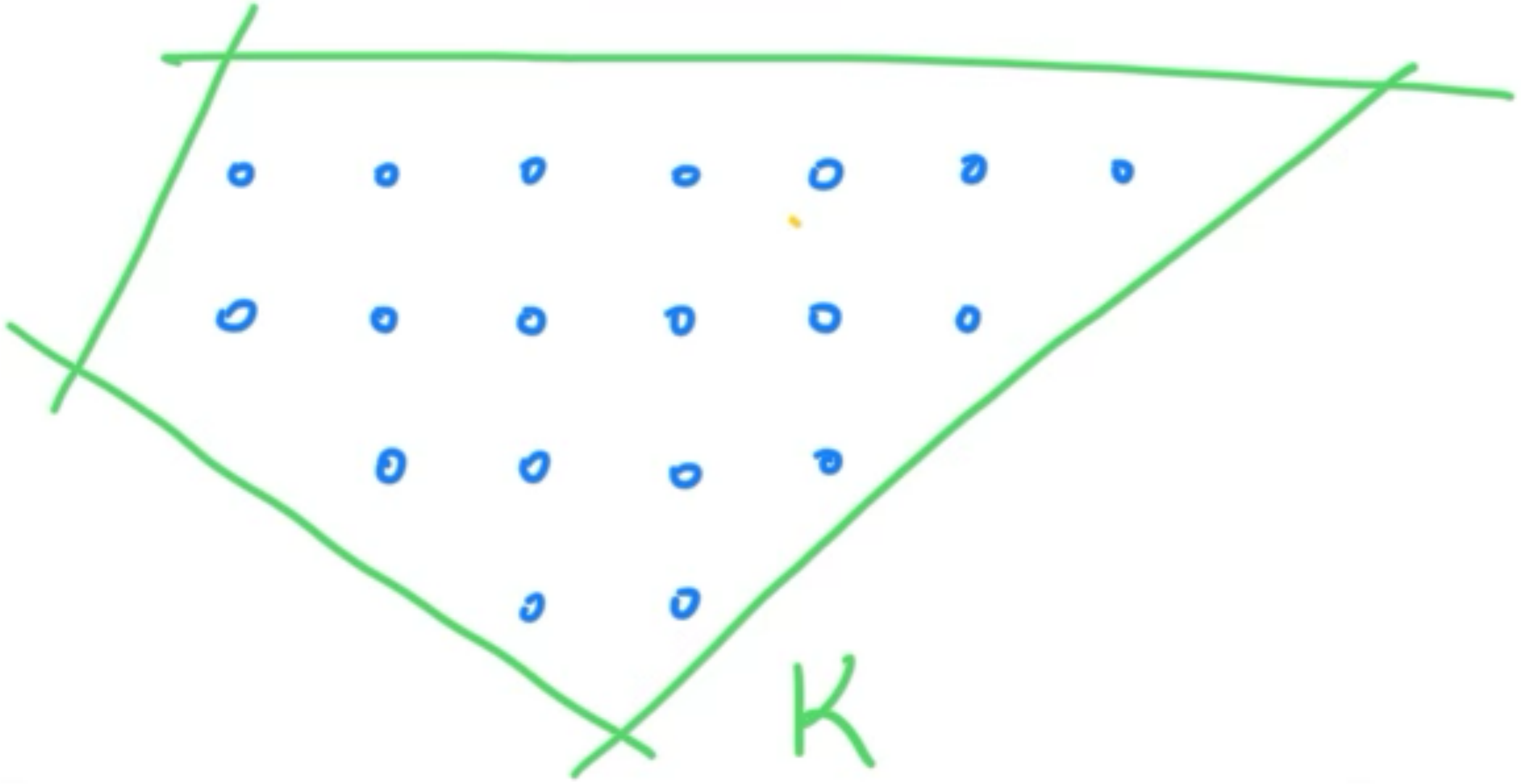
Lenstra-type Integer Programs

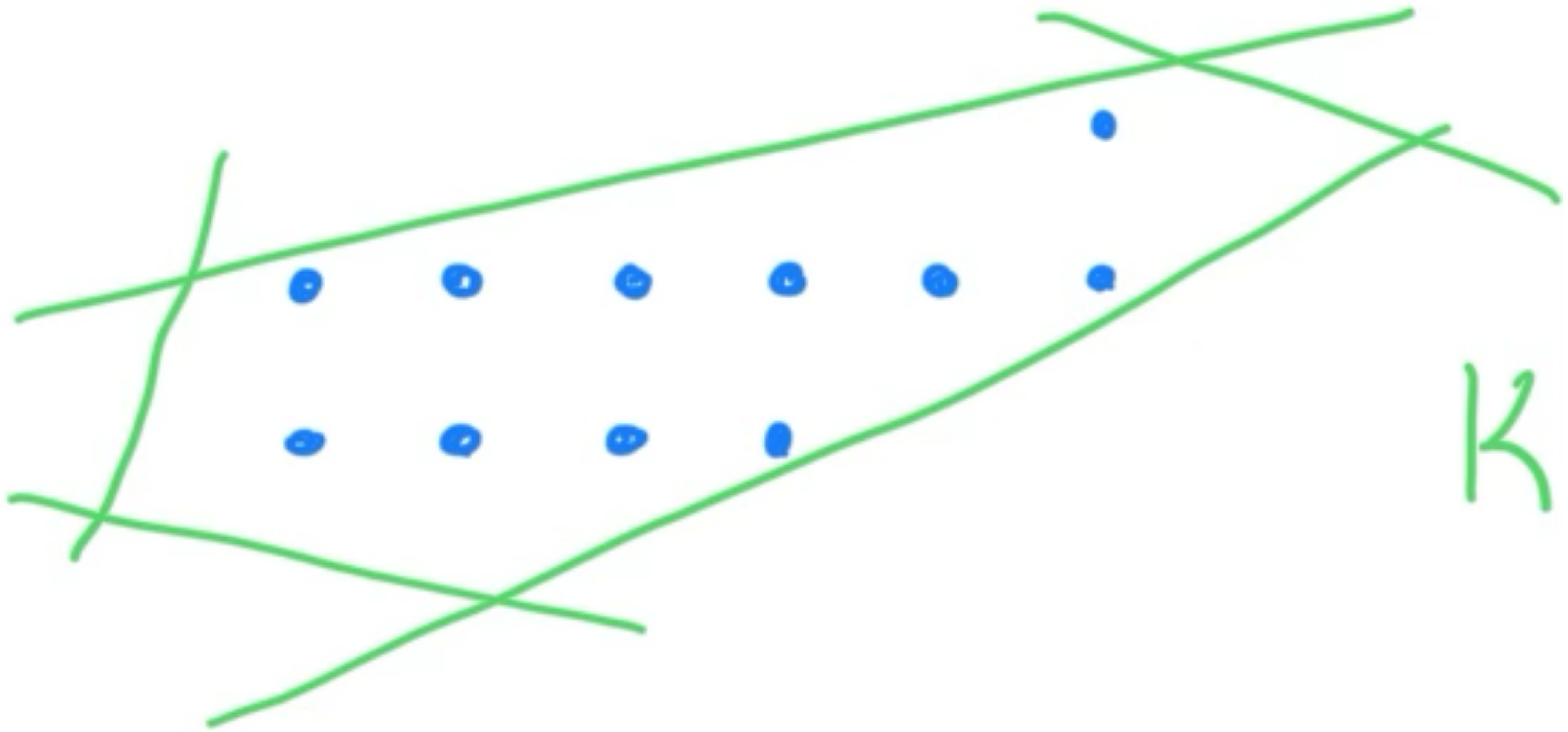


2^{n^3}	Lenstra ‘83
$2^n n^{2.5n}$	Kannan ‘87
$2^n n^n$	Dadush, Peikert, Vempala ‘11, Dadush ‘12
$log(n)^n$	Rothvoss, Reis ‘23

Lenstra-type Integer Programs

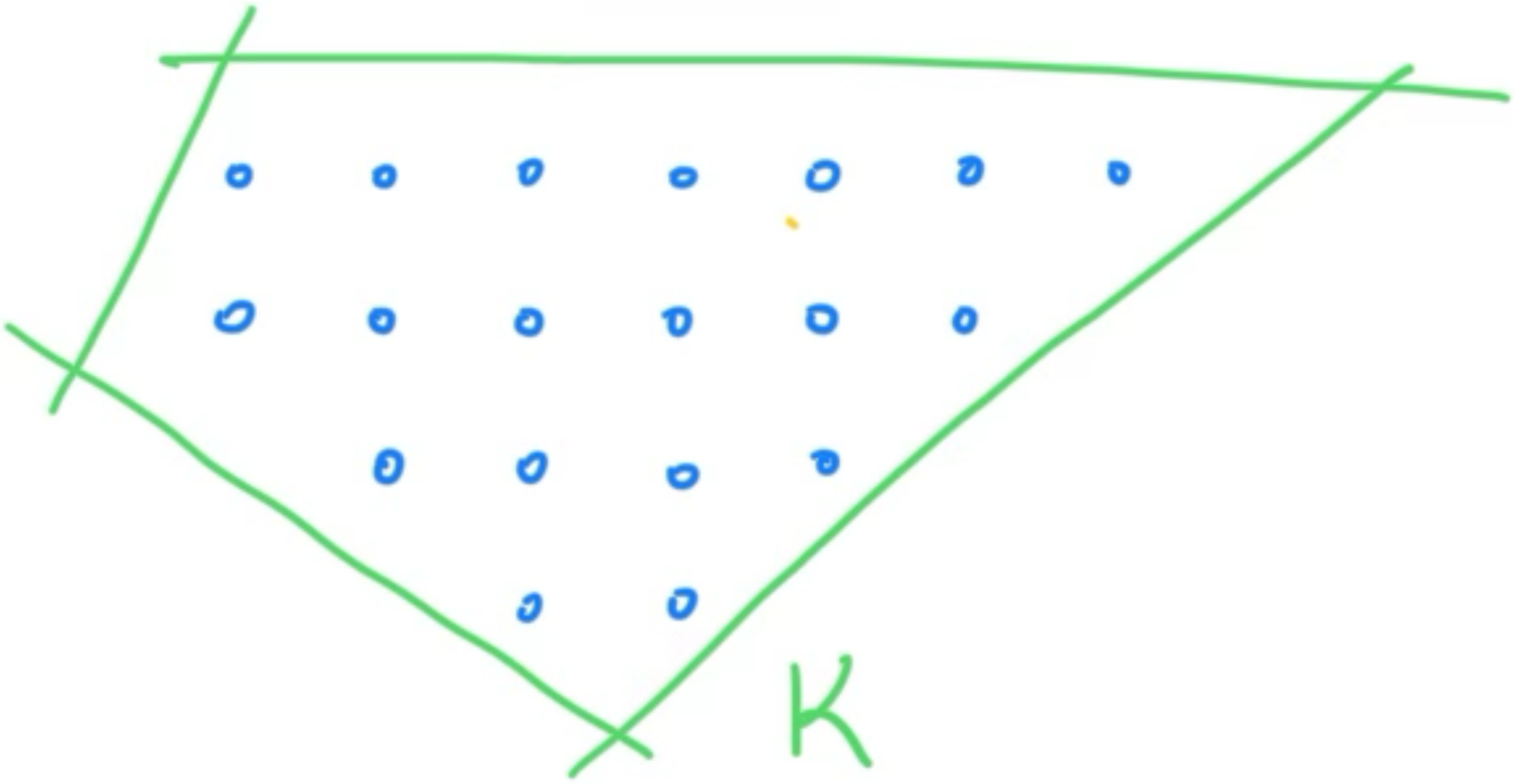
1. Decide whether K is “fat” or “flat”





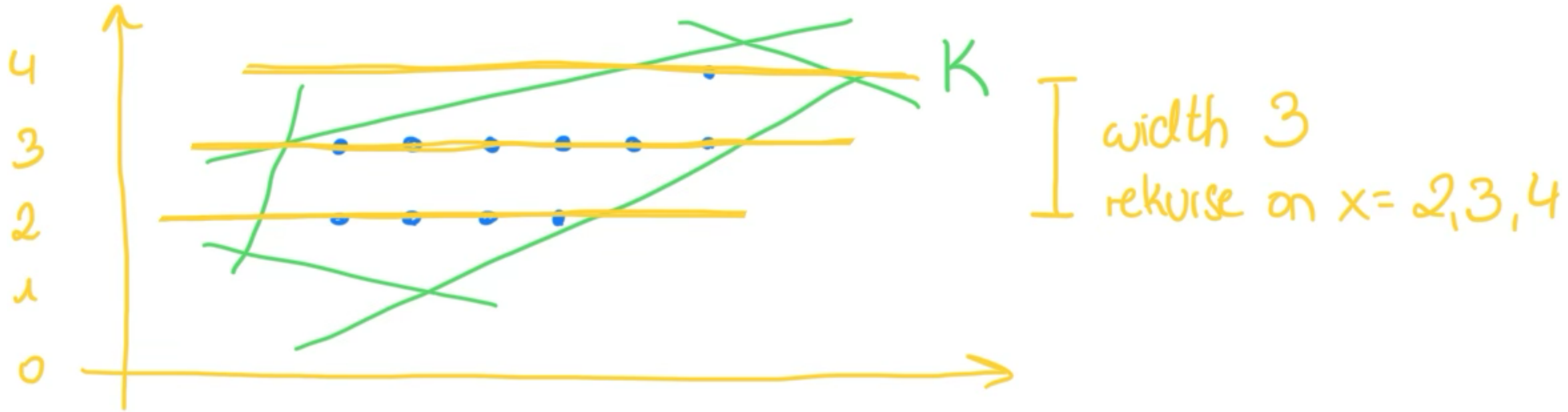
Lenstra-type Integer Programs

1.1. If "fat": easy to find solution

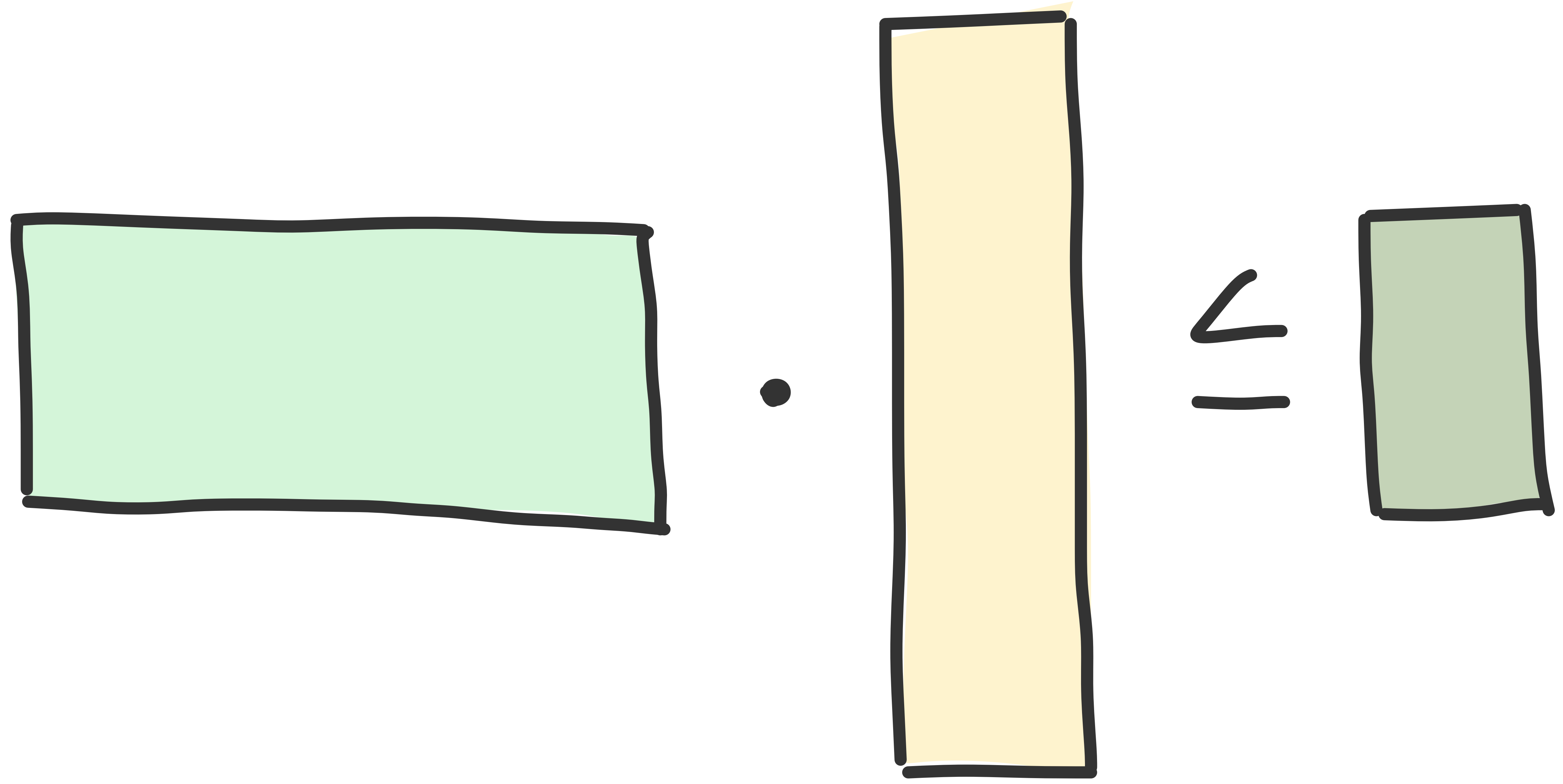


Lenstra-type Integer Programs

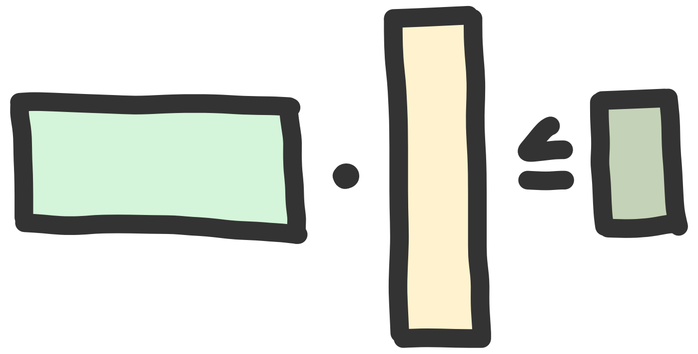
1.2. If “flat”: find flat dimension and decompose



Papademitriou-type Integer Programs



Papademitriou-type Integer Programs



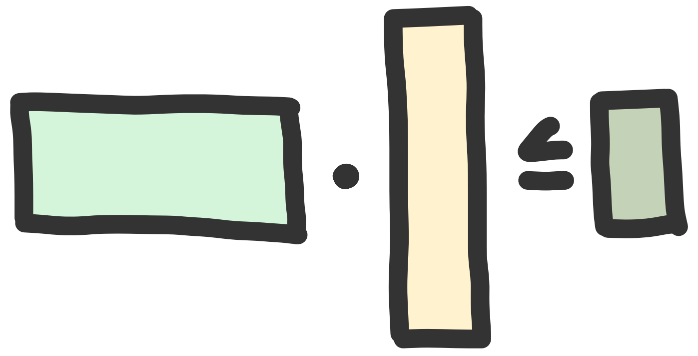
$n^{2m+2} \cdot (m\Delta + m||b||^{\infty})^{(m+1)(2m+1)}$ Papadimitriou '81

$n \cdot (m\Delta)^{2m} \cdot ||b||_1^2$ Eisenbrand, Weismantel '18

$O(nm) \cdot (m\Delta)^{2m} \cdot \log(||b||_{\infty})$ Jansen, Rohwedder '19

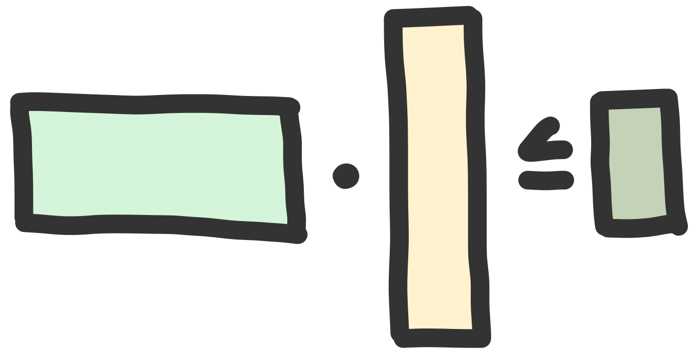
$O(nm) \cdot (\sqrt{m}\Delta)^{2m}$ Jansen Rohwedder '22

Papademitriou-type Integer Programs



Graver elements: inclusion—wise minimal ($\sqsubseteq$) kernel elements

$\sqsubseteq$: $x_i \leq y_i \ \ \forall i$
and sign—compatible

Papademitriou-type Integer Programs 

Graver elements: inclusion—wise minimal ($\sqsubseteq$) kernel elements

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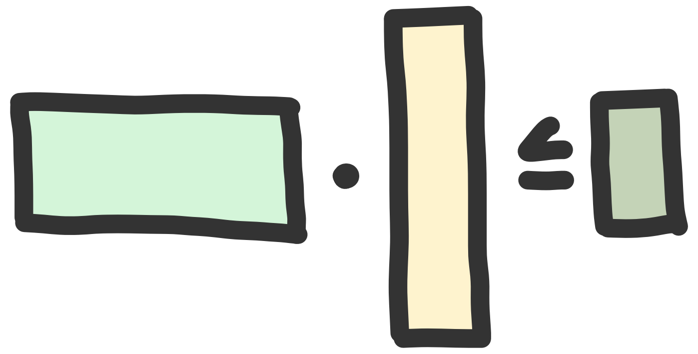
and sign—compatible

$$\begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix} \sqsubseteq \begin{pmatrix} 5 \\ 5 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \not\sqsubseteq \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$$

Papademitriou-type Integer Programs



Graver elements: inclusion—wise minimal (⊑) kernel elements

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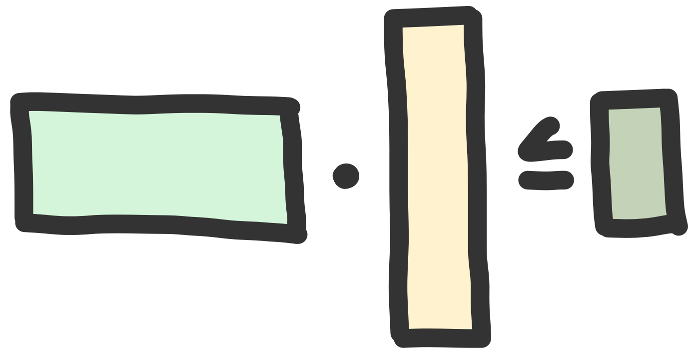
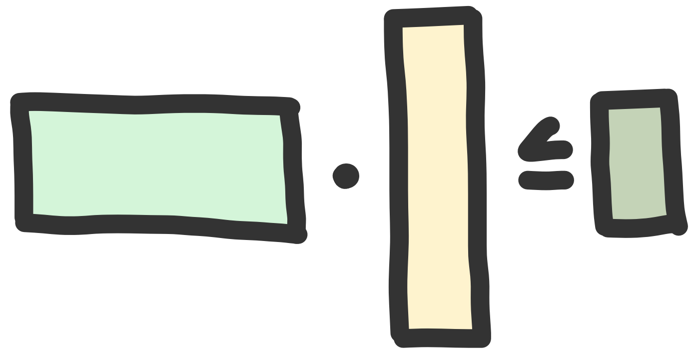
Papademitriou-type Integer Programs



$||g||_\infty \leq \Delta^{f(k)}$ **via Steinitz lemma**

Papademitriou-type Integer Programs



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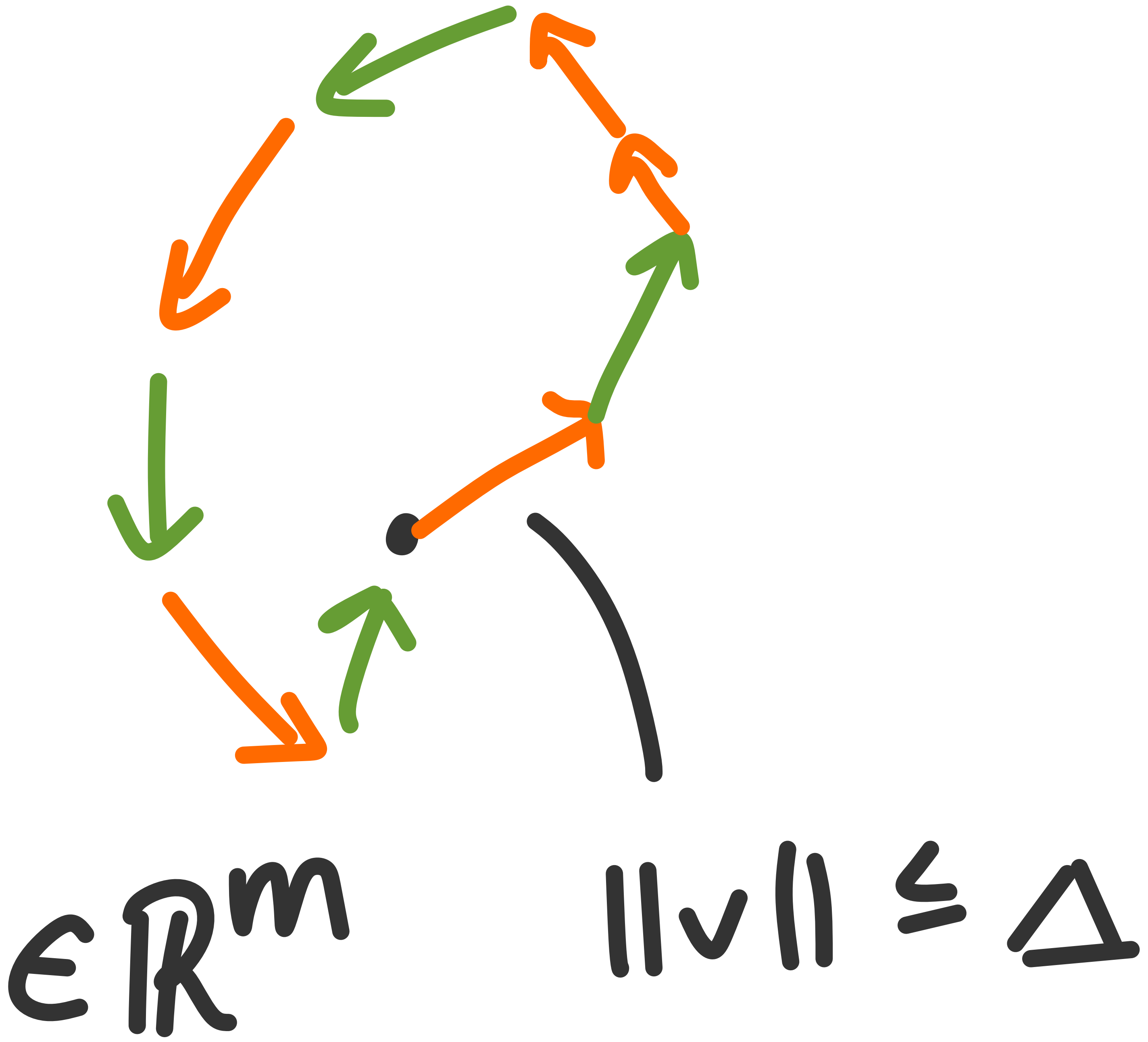
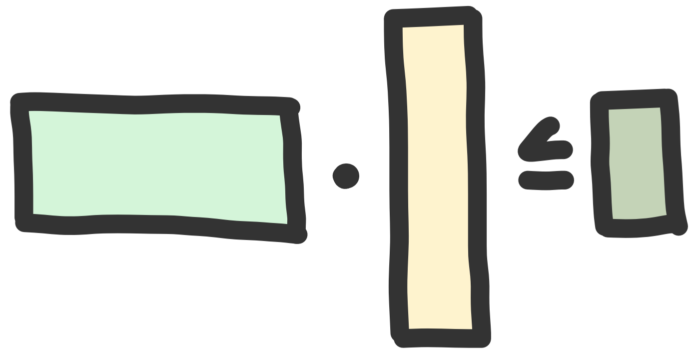


Papademitriou-type Integer Programs



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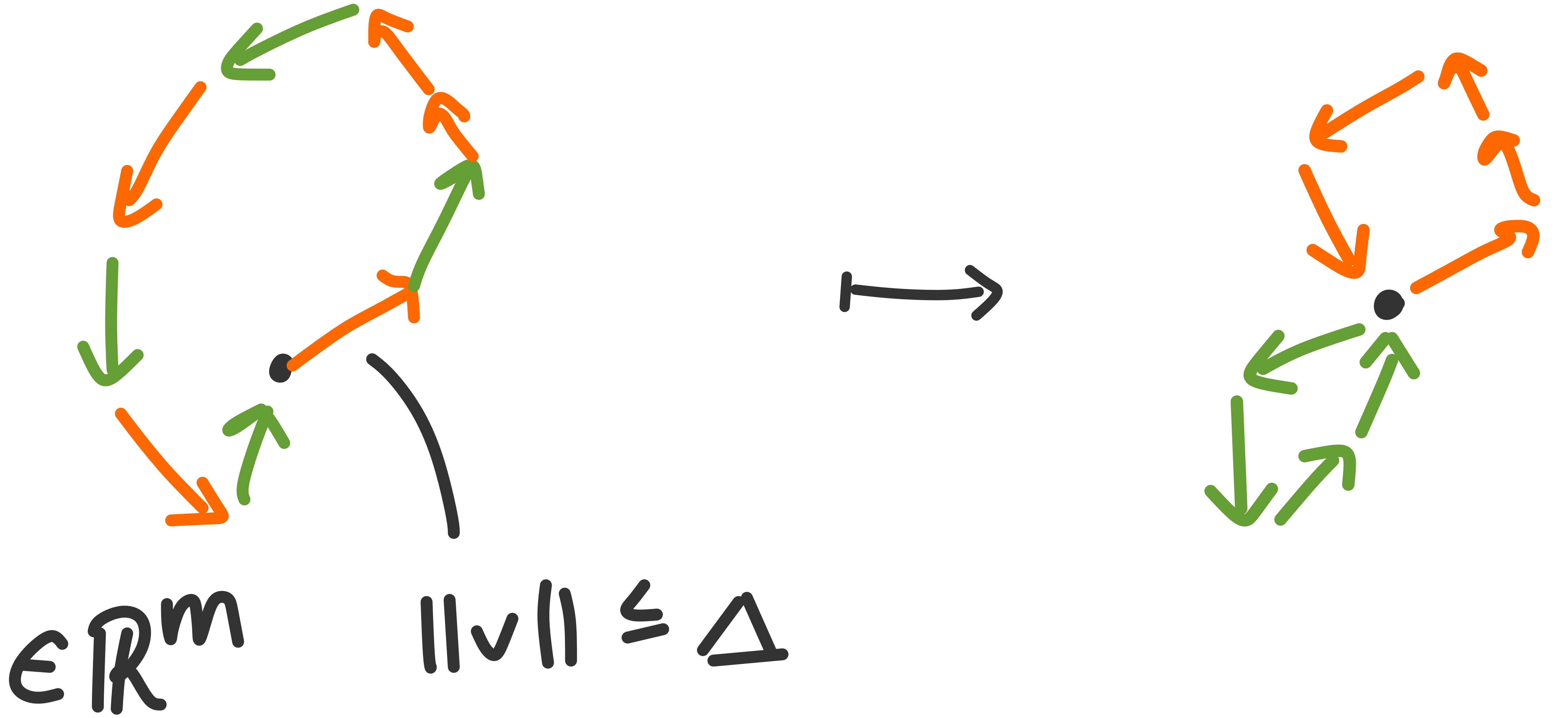
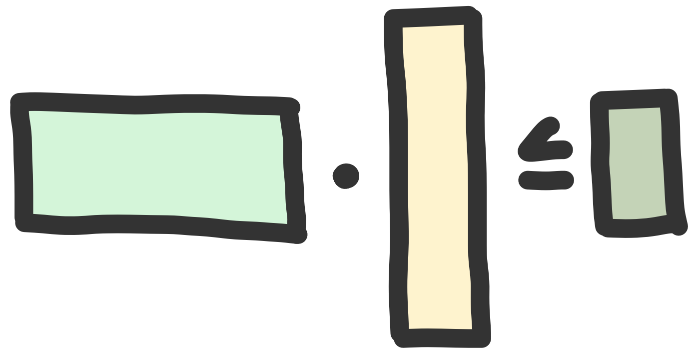


Papademitriou-type Integer Programs



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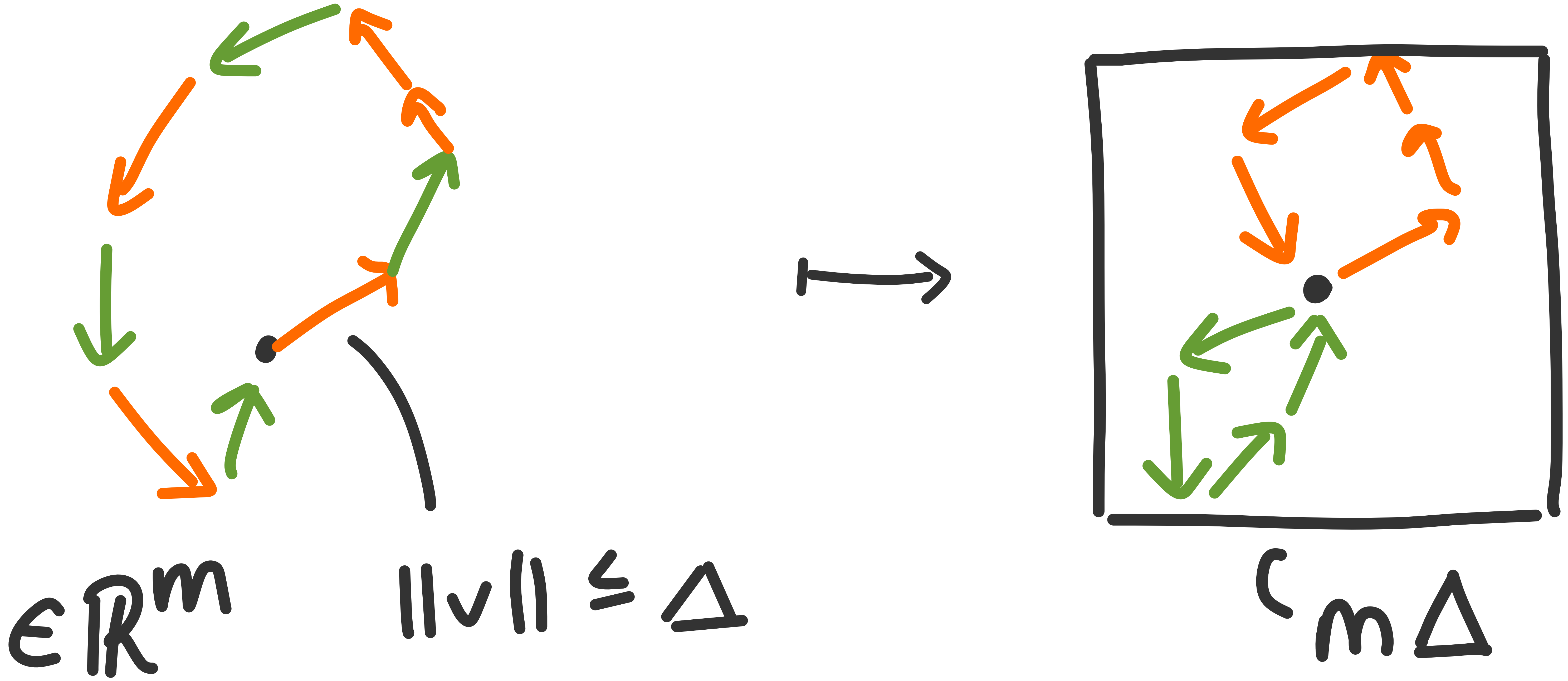

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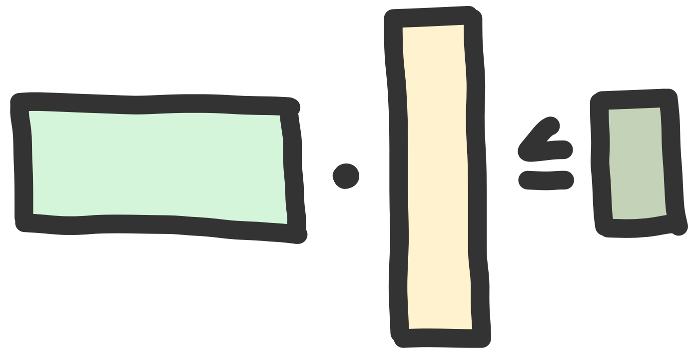

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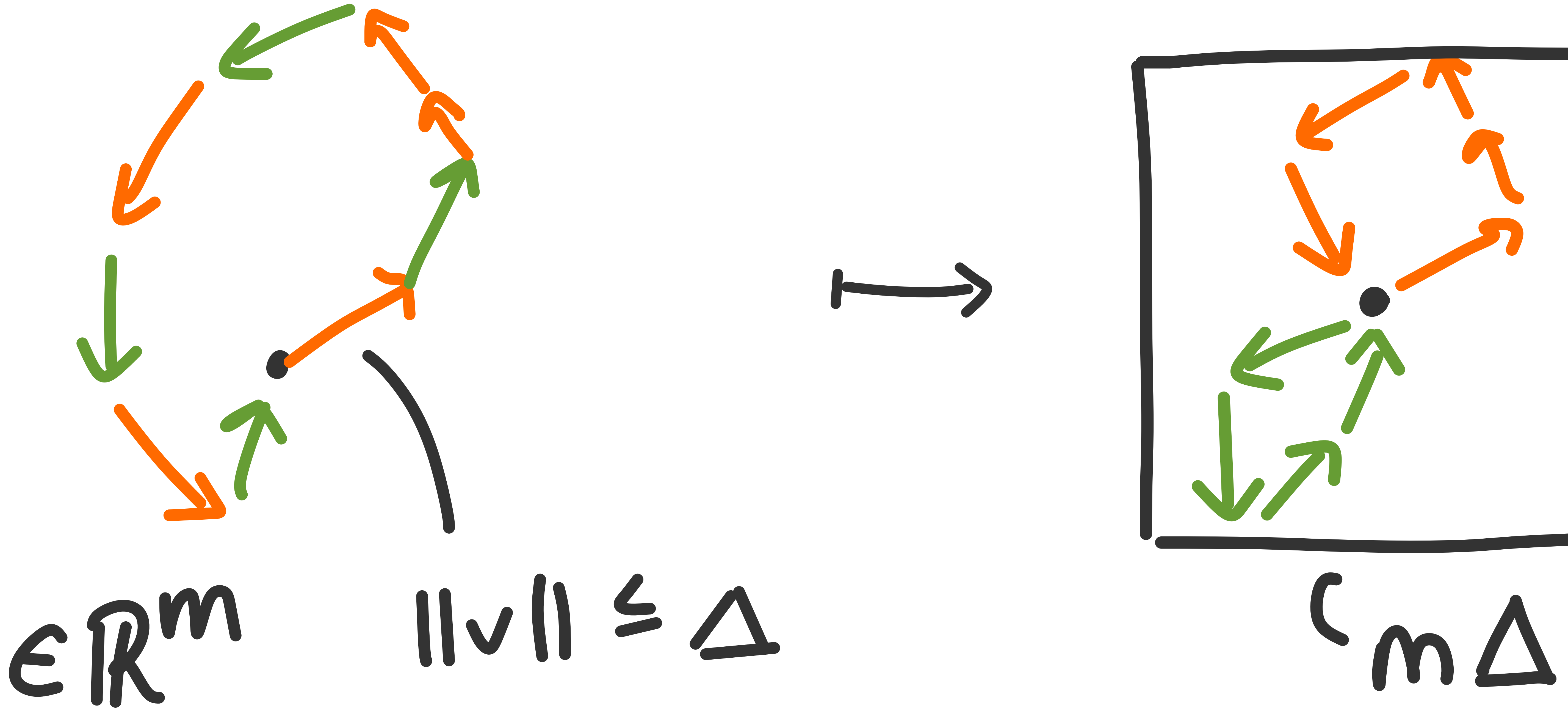

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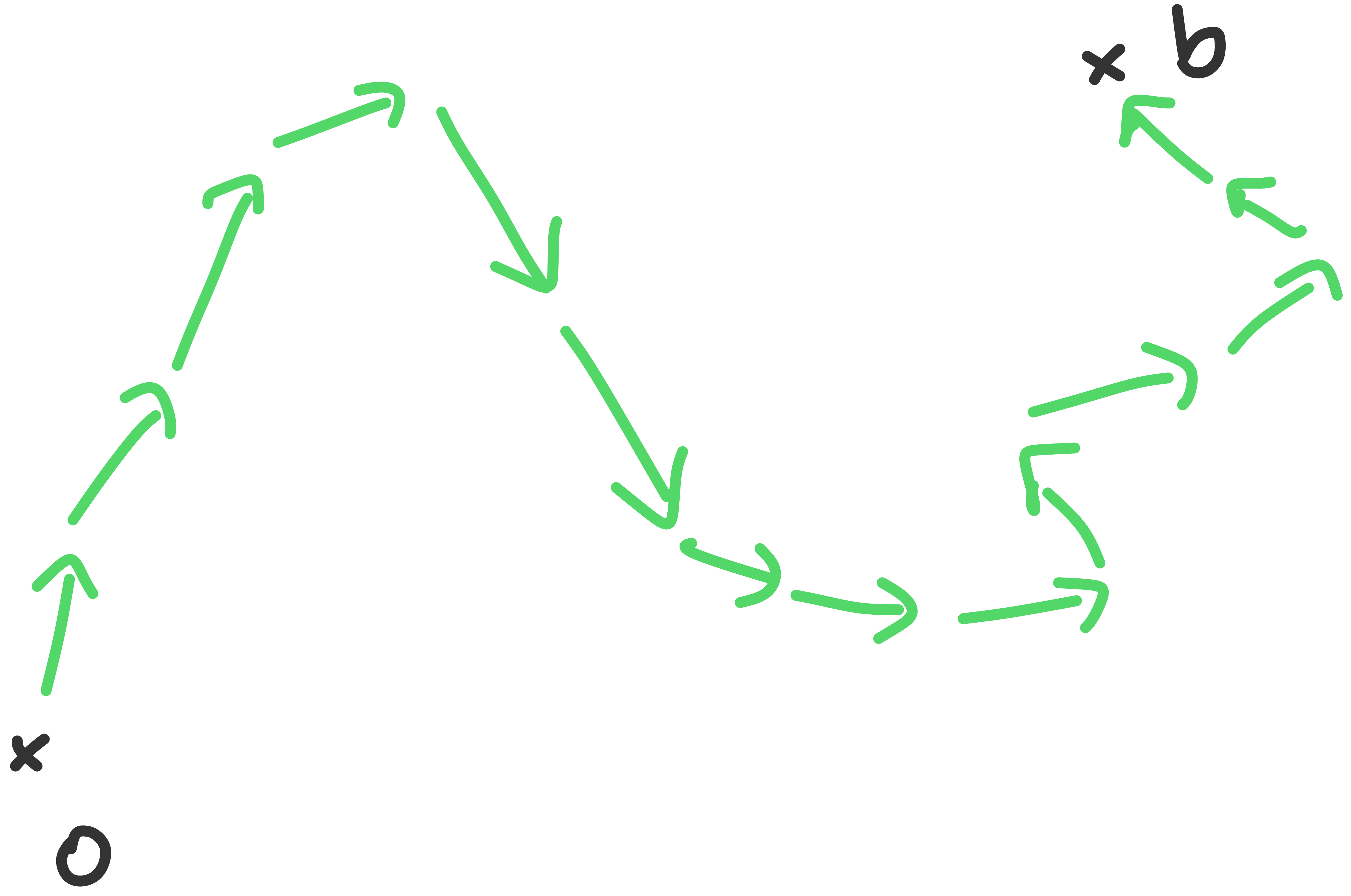

Papademitriou-type Integer Programs