Chance-Constrained Maximum Clique Problem

Shunyu Yao

Department of Business Analytics, University of Iowa

Joint work with Yongzheng Dai, Neng Fan and Pavlo Krokhmal

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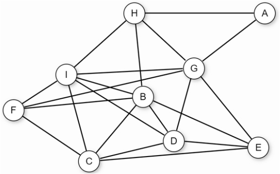
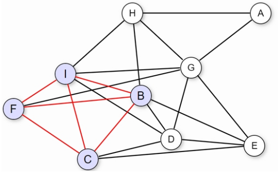
Maximum Clique Problem (MCP)

Clique: a subset of vertices in G where every distinct vertex pair is connected by an edge.

Problem Statement

Given a simple graph G, find a maximum clique (a clique with the largest possible number of vertices) of G.



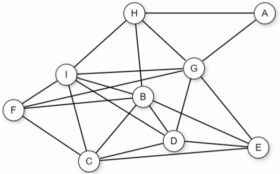


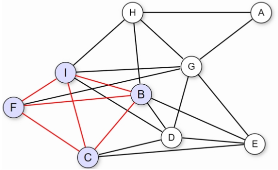
- A maximum clique is {B,C,F,I} with $\omega(G) = 4$, not unique.
- NP-complete, fixed-parameter intractable, hard to approx. within $n^{1-\epsilon}$ if NP $\neq$ ZPP.
- Many applications: social networks, bioinformatics, and computational chemistry, etc.

The cardinality of a maximum clique is called the clique number of G, denoted by $\omega(G)$.

Introduce Uncertainty in Clique Problems

MCP: Given a simple graph G, find a maximum clique of G.





- (**Motivation**) Connections can be inherently uncertain in real-world networks.
- Thus, our goal is to identify a subset that is more likely to induce a larger clique.

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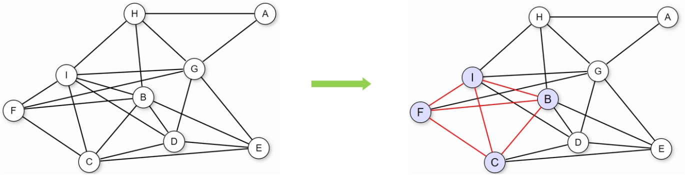

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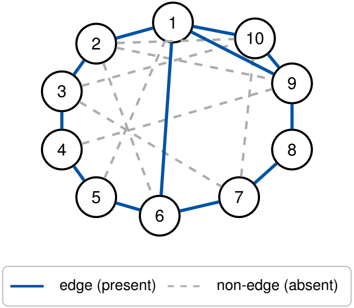
- binary (adjacent/non-adjacent) relationship $\rightarrow$ binary random variable
- $e \in E(G) \rightarrow \mathbb{P}(e$ appears in $G)$
- find a maximum set S in G such that the induced graph $G[S]$ is a clique $\rightarrow$ find a maximum set S such that $\mathbb{P}(G[S]$ is a clique in $G) \geq \theta$ (θ is a predefined parameter)

Random Graphs

The **random graph** $\mathcal{G}(n, (p_{ij}))$ is a probability space over graphs on $V = \{1, 2, \ldots, n\}$, where each potential edge (i, j) appears **independently** with probability p_{ij}.

A graph G in $\mathcal{G}(n, (p_{ij}))$



For every pair of vertices $1 \le i < j \le n$,

$$\mathbb{P}[(i, j) \in \mathcal{G}] = p_{ij}, \qquad 0 \le p_{ij} \le 1,$$

and these events are **mutually independent**.

Special case: Erdős–Rényi random graph $\mathcal{G}(n, p)$

When $p_{ij} = p$ for all $i < j$, each edge appears **independently** with the same probability p.

Probability of a graph in $\mathcal{G}(n, p)$

For a deterministic graph $G = (V(G), E(G))$,

$$\mathbb{P}[\mathcal{G} = G] = p^{|E(G)|}(1 - p)^{\binom{n}{2} - |E(G)|}.$$

Chance-Constrained Maximum Clique Problem (CC-MCP)

Formally, we can define a **Chance-constrained maximum clique problem (CC-MCP)** as follows:

$$\text{[CC-MCP]} \quad \omega_\theta(\mathcal{G}) = \max_{S \subseteq V} \{|S| : \mathbb{P}[S \text{ is a clique in } \mathcal{G}] \geq \theta\}, \tag{1}$$

where $\theta \in [0, 1]$ is a prespecified threshold defined by the user.

- When $\theta = 0$, the problem is trivial and the optimal solution $S = V$.
- When $\theta = 1$, the problem becomes the classical MCP on graph $G^1 = (V, E^1)$, where the edge set $E^1 = \{e \in K_n : p_e = 1\}$.

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Equivalent form:

$$\text{[CC-MCP]} \;\; \omega_\theta(\mathcal{G}) = \max \sum_{i \in V} x_i \tag{2a}$$

$$\text{s.t. } \mathbb{P}[x_i + x_j \leq 1 \; \forall (i,j) \in \bar{\mathcal{G}}] \geq \theta \tag{2b}$$

$$x_i \in \{0, 1\}, \; \forall i \in V \tag{2c}$$

where $\bar{\mathcal{G}}(n, (q_{ij}))$ denotes the complement of $\mathcal{G}(n, (p_{ij}))$ with $\mathbb{P}[(i,j) \in \bar{\mathcal{G}}] = q_{ij}$, $q_{ij} = 1 - p_{ij}, 1 \leq i < j \leq n$.

Mixed-Integer Programming (MIP) Reformulation

Theorem 1

CC-MCP admits the following exact MIP reformulation:

$$\max \sum_{i \in V} x_i \tag{3a}$$

$$\text{s.t.} \sum_{e \in K_n} y_e \log(1/p_e) \leq \log(1/\theta), \tag{3b}$$

$$y_e \leq x_i, y_e \leq x_j \ \forall e = (i,j) \in K_n \tag{3c}$$

$$x_i + x_j - 1 \leq y_e, \ \forall e = (i,j) \in K_n \tag{3d}$$

$$x_i, x_j \in \{0,1\}, 0 \leq y_e \leq 1, \ \forall e = (i,j) \in K_n \tag{3e}$$

- Constraints (3c) – (3e): linearization of the product $y_e := x_i x_j$, where $e = (i,j) \in K_n$.

Mixed-Integer Programming (MIP) Reformulation (cont.)

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Observations

- Note that constraint (3b) is a fractional knapsack problem (FKP) constraint.
  $\rightarrow y_e$ is a continuous variable in constraint (3b).

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Observations

- Note that constraint (3b) is a fractional knapsack problem (FKP) constraint.
- If constraints (3c) & (3d) are removed, the resulting MIP can be solved in a very effective way.

Lagrangian Relaxation

Proposition 1

Constraints (3c) *can be removed from the MIP formulation* (3).

Dualize on the constraints (3d) and consider the Lagrangian relaxation (LR) problem:

$$Z(\lambda) := \max \quad \sum_{i \in V} \sum_{j \in N(i)} x_i \left(\frac{1}{|N(i)|} - \lambda_{ij} \right) + \sum_{e \in K_n} \lambda_e (y_e + 1) \tag{4a}$$

$$\text{s.t.} \quad \sum_{e \in K_n} y_e \log\left(\frac{1}{p_e}\right) \le \log\left(\frac{1}{\theta}\right) \tag{4b}$$

$$x \in \{0,1\}^n, 0 \le y_e \le 1, \forall e \in K_n \tag{4c}$$

Proposition 2

For a fixed $\bar{\lambda}$, $Z(\bar{\lambda})$ *can be computed in* $O(n^2)$ *time.*

Lagrangian Dual (LD) Problem

An optimal value of LR problem (4) is an upper bound for $\omega_\theta(\mathcal{G})$. To find the tightest upper bound over the Lagrangian multiplier vector λ, we consider the following Lagrangian dual (LD) problem:

$$Z_{LD} = \min_{\lambda \geq 0} Z(\lambda)$$

We can apply a standard subgradient algorithm to determine a near-optimal multiplier vector λ, i.e., compute the new multipliers by

$$\lambda^{k+1} := \text{Proj}_{\mathbb{R}^m_+}(\lambda^k - \gamma_k s^k), \tag{5}$$

- s^k is a subgradient of $Z(\lambda)$ at point λ^k
- $\text{Proj}_{\mathbb{R}^m_+}(\cdot)$ is the Euclidean projection on $\mathbb{R}^m_+$
- γ_k is a step length

Lagrangian cuts

Observation

For any multiplier $\tilde{\lambda}$, an optimal value of LR provides a valid upper bound for maximization problems, i.e., $\omega_\theta(\mathcal{G}) \leq Z(\tilde{\lambda})$.

Proposition 3

For any fixed multiplier $\tilde{\lambda}$, the Lagrangian cut

$$\sum_{i \in V} \left(\sum_{j:(i,j) \in E} \tilde{\lambda}_{ij} \right) x_i \leq \sum_{(i,j) \in E} \tilde{\lambda}_{ij} + \underbrace{\max_{\substack{0 \leq y \leq 1, \\ \sum w_{ij} y_{ij} \leq w'}} \sum_{(i,j)} \tilde{\lambda}_{ij} y_{ij}}_{\text{FKP problem}} . \tag{6}$$

is valid for the problem (3).

Delayed Constraint Generation (DCG)

We start by solving the following relaxed problem:

$$\max\left\{\sum_{i\in V} x_i : \text{(3b)}, \text{(3e)}, x_i + x_j - 1 \le y_e,\ \forall t, e = (i,j) \in E^t \subseteq K_n\right\}, \tag{7}$$

where E^t represents the edge set that violates constraints (3d) during the tth iteration.

- (Feasibility Check) if each edge $e \in K_n$ satisfies the CC-MCP constraint (3d), we have arrived at an optimal solution.
- (Cut Generation) o.w., add the Lagrangian cut (6) and constraint $x_i + x_j - 1 \le y_e$ to the relaxed problem and repeat.

The DCG algorithm will converge to an optimal solution in at most $O(n^2)$ iterations due to $\binom{n}{2}$ constraints (3d).

Experiment Settings

- Networks:
  - Complete graphs $n \in \{50, 100, 150, 200\}$
  - Real-world networks
- Parameter settings:
  - Threshold $\theta_i \in \{0.3, 0.5, 0.7, 0.9\}$
  - Half deterministic edges; remaining: $p_{ij} \sim$ Uniform(0.8,1)
  - Time limit - 1h
- Methods
  - DP (dynamic programming, Miao et al., 2014[1])
  - Default (solved directly by Gurobi)
  - DCG (w/o LCs)
  - DCG+LC (DCG with LCs)

[1] Miao, Zhuqi, Balabhaskar Balasundaram, and Eduardo L. Pasiliao. ”An exact algorithm for the maximum probabilistic clique problem.” Journal of Combinatorial Optimization 28.1 (2014): 105-120.

Numerical Results for Complete Graphs

Parameters		DP		Default			DCG			DCG+LC		
n	θ	Val	Time	Val	Time	Gap(%)	Val	Time	Gap(%)	Val	Time	Gap(%)
50	0.3	13	309.78	13	17.30	0.00	**13**	**1.40**	**0.00**	13	41.66	0.00
50	0.5	12	16.63	12	8.38	0.00	**12**	**0.95**	**0.00**	12	19.25	0.00
50	0.7	11	1.60	11	3.88	0.00	**11**	**0.64**	**0.00**	11	0.64	0.00
50	0.9	**9**	**0.04**	9	1.61	0.00	9	0.66	0.00	9	0.85	0.00
100	0.3	14	>1h	14	>1h	71.43	15	>1h	26.67	**16**	**458.78**	**0.00**
100	0.5	14	>1h	14	>1h	42.86	**14**	**766.08**	**0.00**	14	1875.38	0.00
100	0.7	**13**	**111.03**	13	>1h	15.38	13	657.10	0.00	13	893.55	0.00
100	0.9	**11**	**2.23**	11	101.42	0.00	11	27.89	0.00	11	35.84	0.00
150	0.3	**16**	>1h	14	>1h	185.71	15	>1h	193.33	**16**	>1h	**50.00**
150	0.5	15	>1h	13	>1h	169.23	15	3580.43	0.00	**15**	**1418.37**	**0.00**
150	0.7	13	>1h	12	>1h	158.33	14	1473.15	0.00	**14**	**1295.66**	**0.00**
150	0.9	**12**	**32.52**	12	1268.06	0.00	12	189.98	0.00	12	207.06	0.00
200	0.3	14	>1h	12	>1h	450.00	**17**	>1h	**182.35**	16	>1h	343.75
200	0.5	14	>1h	13	>1h	276.92	14	>1h	335.71	**15**	>1h	280.00
200	0.7	**14**	>1h	13	>1h	253.85	13	>1h	353.85	**14**	>1h	335.71
200	0.9	**13**	**141.27**	12	>1h	100.00	13	953.61	0.00	13	1215.24	0.00

Numerical Results for Real-World Networks

Parameters		DP		Default			DCG			DCG+LC		
Graph	θ	Val	Time	Val	Time	Gap(%)	Val	Time	Gap(%)	Val	Time	Gap(%)
C125-9	0.3	14	$>$ 1h	14	$>$ 1h	50.00	15	1615.34	0.00	**15**	**871.92**	**0.00**
C125-9	0.5	13	3390.33	13	$>$ 1h	30.77	13	483.12	0.00	**13**	**319.29**	**0.00**
C125-9	0.7	**12**	**144.95**	12	1970.35	0.00	12	253.14	0.00	12	177.98	0.00
C125-9	0.9	**11**	**2.59**	11	252.39	0.00	11	54.15	0.00	11	51.99	0.00
c-fat200-5	0.3	15	$>$ 1h	15	1974.23	0.00	15	414.03	0.00	**15**	**390.95**	**0.00**
c-fat200-5	0.5	13	695.91	13	890.77	0.00	13	298.68	0.00	**13**	**281.87**	**0.00**
c-fat200-5	0.7	**12**	**38.12**	12	868.06	0.00	12	234.63	0.00	12	211.34	0.00
c-fat200-5	0.9	**10**	**1.11**	10	1129.48	0.00	10	59.93	0.00	10	41.72	0.00
gen200-p0-9-55	0.3	14	$>$ 1h	15	$>$ 1h	153.33	15	$>$ 1h	66.67	**15**	$>$ 1h	**53.33**
gen200-p0-9-55	0.5	14	$>$ 1h	13	$>$ 1h	161.54	15	2677.23	0.00	**15**	**2167.80**	**0.00**
gen200-p0-9-55	0.7	14	2704.27	12	$>$ 1h	158.33	**14**	**922.37**	**0.00**	14	961.33	0.00
gen200-p0-9-55	0.9	**12**	**43.81**	11	$>$ 1h	81.82	12	366.06	0.00	12	381.22	0.00
san200-0-9-1	0.3	14	$>$ 1h	14	$>$ 1h	178.57	**16**	$>$ 1h	**50.00**	**16**	$>$ 1h	**50.00**
san200-0-9-1	0.5	13	$>$ 1h	13	$>$ 1h	176.92	15	3111.07	0.00	**15**	**2847.76**	**0.00**
san200-0-9-1	0.7	13	$>$ 1h	12	$>$ 1h	141.67	13	1116.00	0.00	**13**	**1037.27**	**0.00**
san200-0-9-1	0.9	**11**	**56.83**	10	$>$ 1h	110.00	11	663.42	0.00	11	381.16	0.00

Thanks for your attention!

Shunyu Yao
shunyu-yao@uiowa.edu