Sensitivity analysis for linear changes of the constraint matrix of an LP and an MIP

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Based on a joint work with

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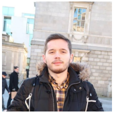

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Sensitivity analysis

Consider a Linear Program

$$\begin{aligned}
\min \quad & c_1x_1 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
& \vdots \\
& a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \\
& x_1, \ \ldots, \ x_n \geq 0
\end{aligned}$$

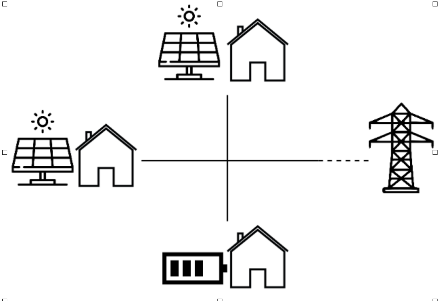
Sensitivity analysis

Consider a Linear Program

$$\begin{array}{rl} \min & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n = b_1 + \lambda \\ & \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \\ & x_1, \ \ldots, \ x_n \geq 0 \end{array}$$

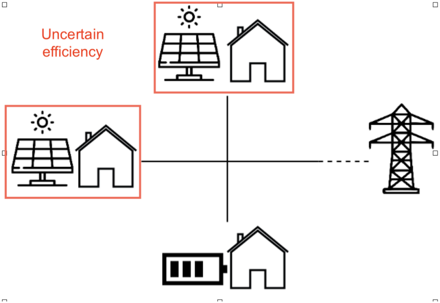
- Modify one coefficient in the right-hand-side
- Gives a nice piecewise linear convex function of λ

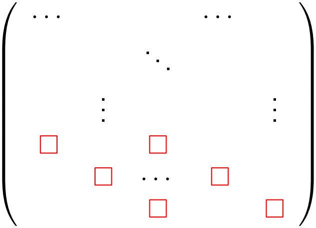
Sometimes one parameter influences several coefficients simultaneously



$$\begin{pmatrix} \cdots & & & \cdots \\ & & \ddots & \\ & \vdots & & \vdots \\ & & \cdots & \end{pmatrix}$$

Sometimes one parameter influences several coefficients simultaneously





The formalization

$$\begin{aligned} f(\lambda) = \min\ & c^T x \\ \text{subject to}\ & A_1 x \leq b_1 \\ & (A_2 + \lambda D)x \leq b_2 \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{aligned}$$

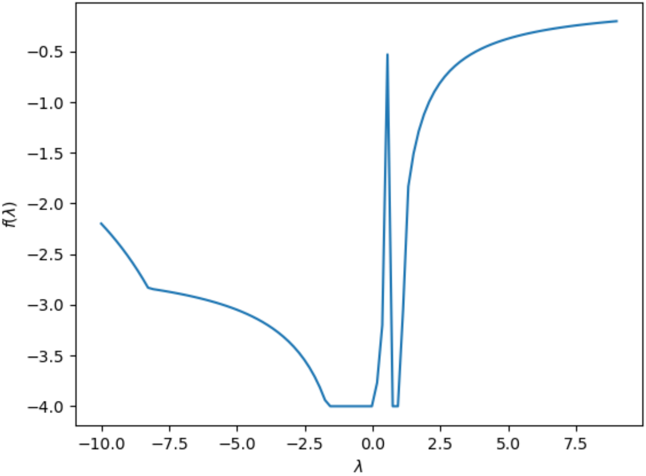
for $\lambda \in [\lambda_1, \lambda_2]$.

Questions:

- Shape of $f(\lambda)$ depending on D ?
- What is the complexity of computing $\min_{\lambda \in [\lambda_1, \lambda_2]} f(\lambda)$ or $\max_{\lambda \in [\lambda_1, \lambda_2]} f(\lambda)$?

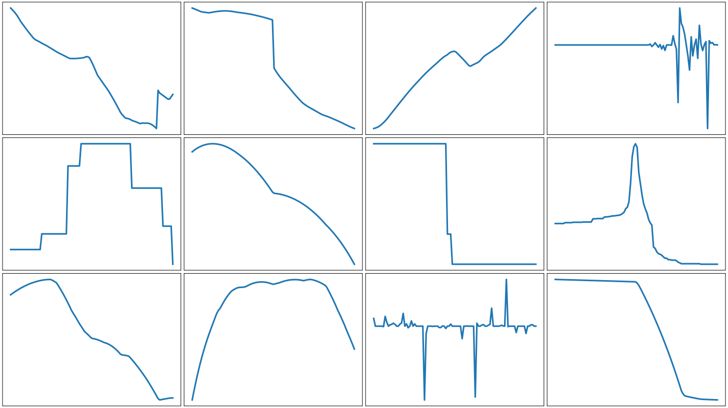
Bad news : it can be ugly

Simple 2-variable continuous problem



More examples

Some problems from our library (both LP and MIP)



As expected, optimizing is hard in general

Theorem

For a purely continuous problem, finding

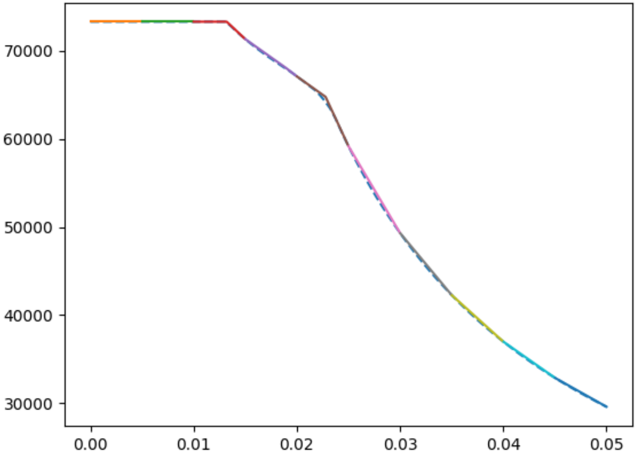
$$\begin{array}{ccc} \min\ f(\lambda) & \textit{or} & \max\ f(\lambda) \\ \lambda \in [\lambda_1, \lambda_2] & & \lambda \in [\lambda_1, \lambda_2] \end{array}$$

is NP-hard.

A special tractable case

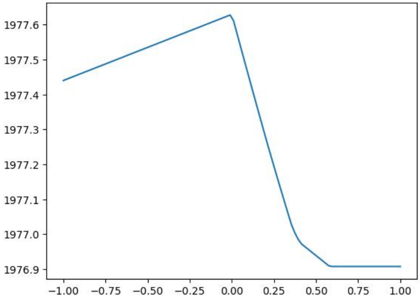
If $D \geq 0$ or $D \leq 0$, $f(\lambda)$ is monotone both for LPs and MIPs.

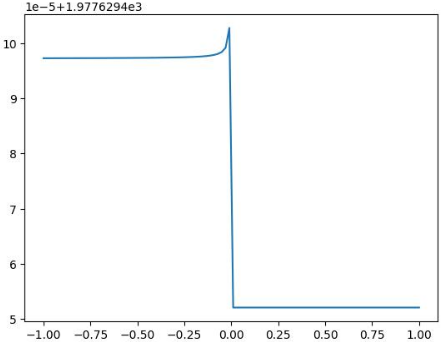
The extrema are either in λ_1 or in λ_2 $\rightarrow$ polytime for LP!



D is just one row or one column

The proof of NP-hardness requires 2 nonzero columns in D.





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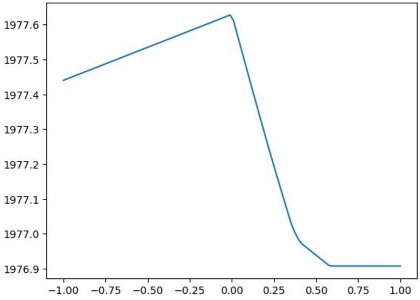
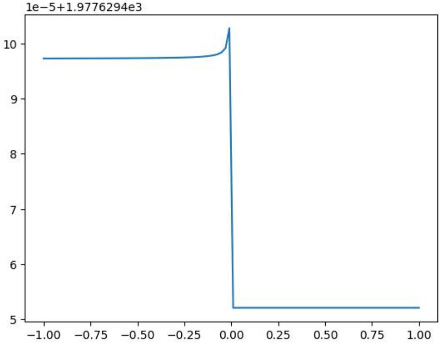
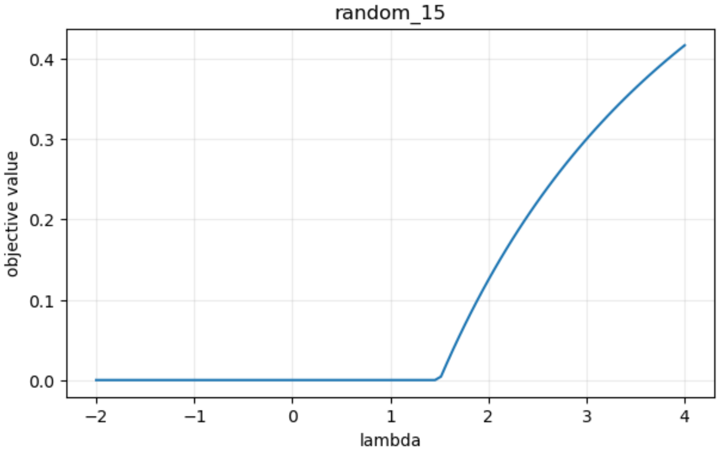




For LP, D with one nonzero row (or column) $\rightarrow f(\lambda)$ quasi-concave (or quasi-convex).
Finding max or min can be done in polynomial time.

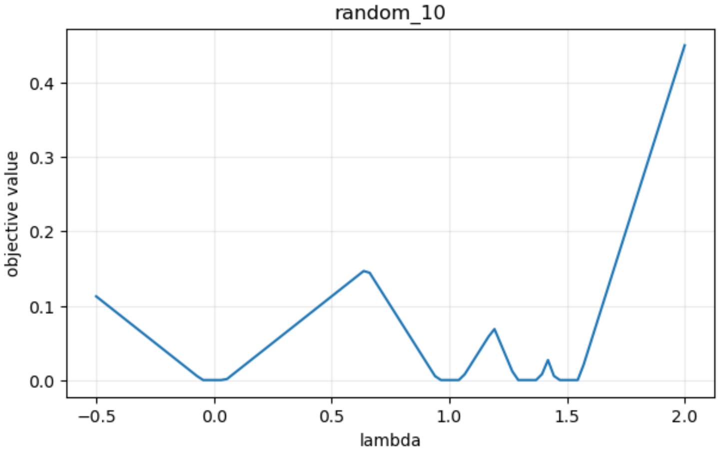
Extending to MIP

If D is one nonzero row, the proof of quasi-concavity extends to MIP.



Extending to MIP

If D is one nonzero column, the proof of quasi-convexity fails to extend.



In MIP, there is an assymetry between 1 row and 1 column.
Modifying one column is much harder than modifying one row.

Looking at the optimal variables

Consider a modification of the cost c

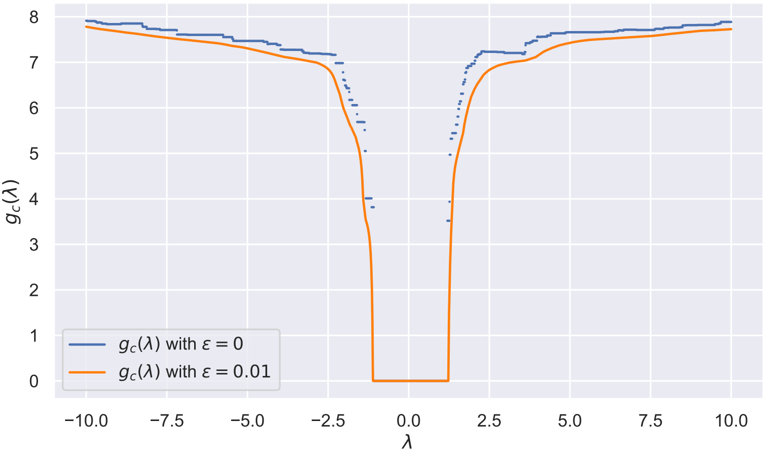
$$
\begin{aligned}
f(\lambda) = \min\ & (c + \lambda d)^T x \\
\text{s.t.}\ \ & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

and check for a secundary objective over the optimal variables

$$
\begin{aligned}
g_c^{\epsilon}(\lambda) = \min\ & g^T x \\
\text{s.t.}\ \ & c^T x \leq f(\lambda) + \epsilon \\
& Ax \leq b \\
& x \geq 0
\end{aligned}
$$

g_c is not necessarily continuous

For $\epsilon = 0$, $g_c(\lambda)$ is not continuous.
For all $\epsilon > 0$, you recover continuity.



3 papers on the topic

Efficiently computing bounds of $f(\lambda)$

Complexity questions

Secundary objective on the optimal variables