Polyhedral Structures for Parametric Linear Programs

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MIXED INTEGER PROGRAMMING WORKSHOP

Motivation

- Parametric LP studies how optimal solutions change with the parameters.
- Main computational challenge: the number of critical regions.
- We derive upper bounds via polyhedral structures.
- For fixed parameter dimension, this gives tractable cases.

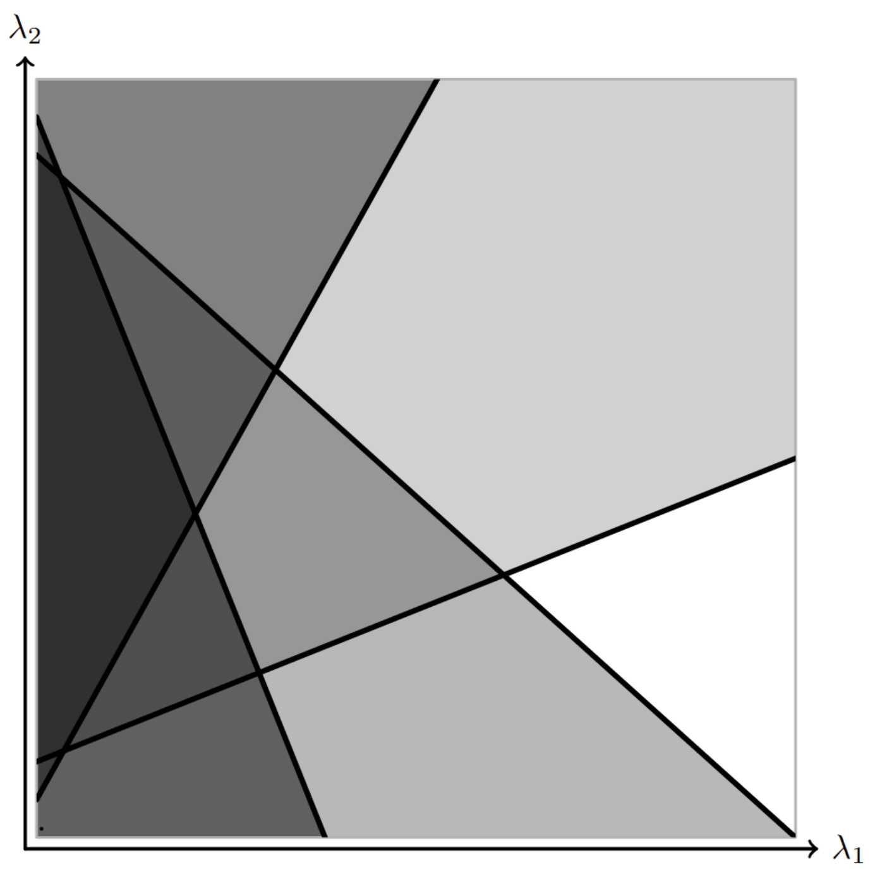
Problem setting

Let $\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell \in \mathbb{R}^n$ and let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polytope.
The parametric linear optimization problem is the function
$\varphi: \Lambda \subseteq \mathbb{R}^\ell \longrightarrow \mathbb{R}^n$ defined by

$$\varphi(\boldsymbol{\lambda}) = \max\left\{\left(\boldsymbol{c}_0 + \sum_{i=1}^{\ell} \lambda_i \boldsymbol{c}_i\right)^\top \boldsymbol{x} : \boldsymbol{x} \in \mathcal{P}\right\}$$

- As λ varies, different vertices of $\mathcal{P}$ may become optimal.
- Each critical region is a set of parameters with the same optimal vertex.
- The objective vector depends affinely on the parameters:

$$\boldsymbol{c}(\boldsymbol{\lambda}) = \boldsymbol{c}_0 + \textstyle\sum_{i=1}^{\ell} \lambda_i \boldsymbol{c}_i$$



Example: $\mathcal{P} = [0,1]^4, \quad \boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2_{\geq 0}$

$$\boldsymbol{c}_0 = \begin{pmatrix} -1 \\ 9 \\ -0.5 \\ 9.5 \end{pmatrix}, \quad \boldsymbol{c}_1 = \begin{pmatrix} -0.4 \\ -0.9 \\ -1.8 \\ -2.5 \end{pmatrix}, \quad \boldsymbol{c}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned}\varphi(\boldsymbol{\lambda}) &= \max\{\boldsymbol{c}(\boldsymbol{\lambda})^\top \boldsymbol{x} : \boldsymbol{x} \in \mathcal{P}\} \\ &= \max\{(\boldsymbol{c}_0 + \lambda_1 \boldsymbol{c}_1 + \lambda_2 \boldsymbol{c}_2)^\top \boldsymbol{x} : \boldsymbol{x} \in \mathcal{P}\}\end{aligned}$$

Why **Critical Regions** Matter?

- The number of critical regions drives the complexity of parametric LP.
- Exponential growth is possible in general.
- Structural bounds are therefore key to tractability.

Our Goal

Which properties can affect the number of critical regions?

- Polyhedral properties of the feasible set.
- Structural properties of the parametric objective.

Related work

Parametric-LP algorithms (e.g. Gusfield 1983) are output-polynomial in oracle calls.

Most upper bound on the nb of critical regions exist for the case $\ell = 1$ for:

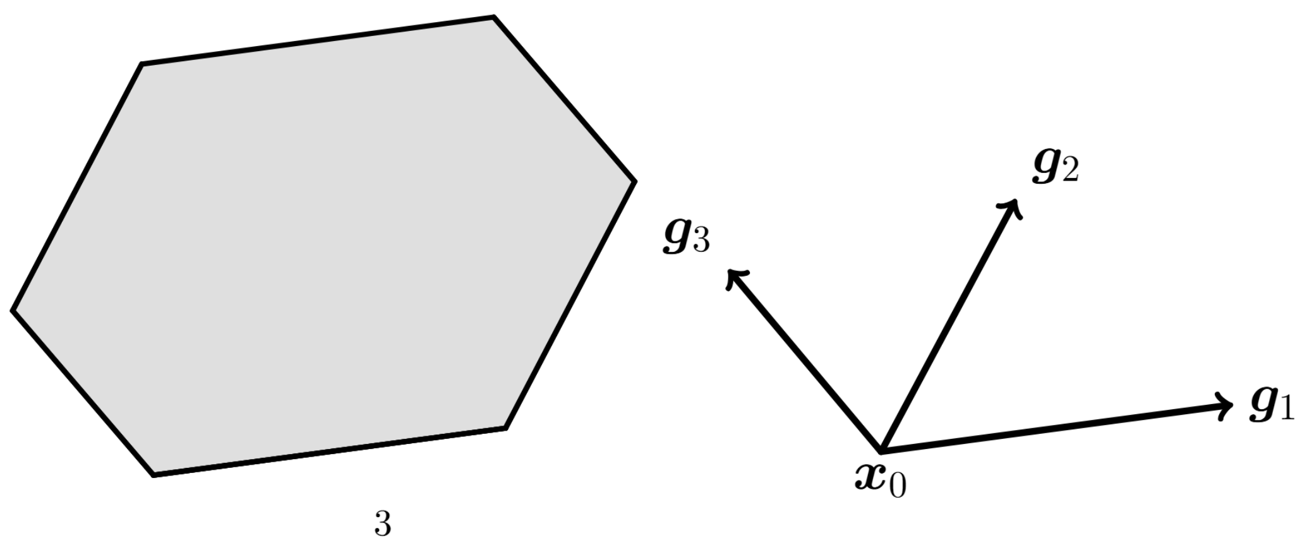
- special combinatorial problems (Gusfield 1980, Carstensen 1983, Correa et al. 2017)
- matroid polytopes and hypercube (Hrubeš and Yehudayoff, 2021)

Polytopes with few edge directions

Zonotopes and hyperplane arrangements

A **zonotope** $\mathcal{Z} \in \mathbb{R}^n$ is a Minkowski sum of finitely many line segments:

$$\mathcal{Z} = \boldsymbol{x}_0 + [\boldsymbol{0}, \boldsymbol{g}_1] + \cdots + [\boldsymbol{0}, \boldsymbol{g}_k].$$



$$\mathcal{Z} = \boldsymbol{x}_0 + \sum_{j=1}^{3} [0, \boldsymbol{g}_j]$$

For a zonotope $\mathcal{Z}$ generated by $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k$:

- The hyperplanes $\{\boldsymbol{\lambda}: \boldsymbol{c}(\boldsymbol{\lambda})^\top \boldsymbol{g}_j = 0\}$ partition the parameter space.
- Optimal vertex depends only on the sign pattern of $\boldsymbol{c}(\boldsymbol{\lambda})^\top \boldsymbol{g}_1, \ldots, \boldsymbol{c}(\boldsymbol{\lambda})^\top \boldsymbol{g}_k$.
- We only care about the cases : $\{\boldsymbol{\lambda}: \boldsymbol{c}(\boldsymbol{\lambda})^\top \boldsymbol{g}_j = 0\} \cap \operatorname{ri} \Lambda \neq \emptyset$
- **Tight Upper Bound**: $\sum_{i=0}^{\ell} \binom{k}{i}$ (via hyperplane arrangement theorem)

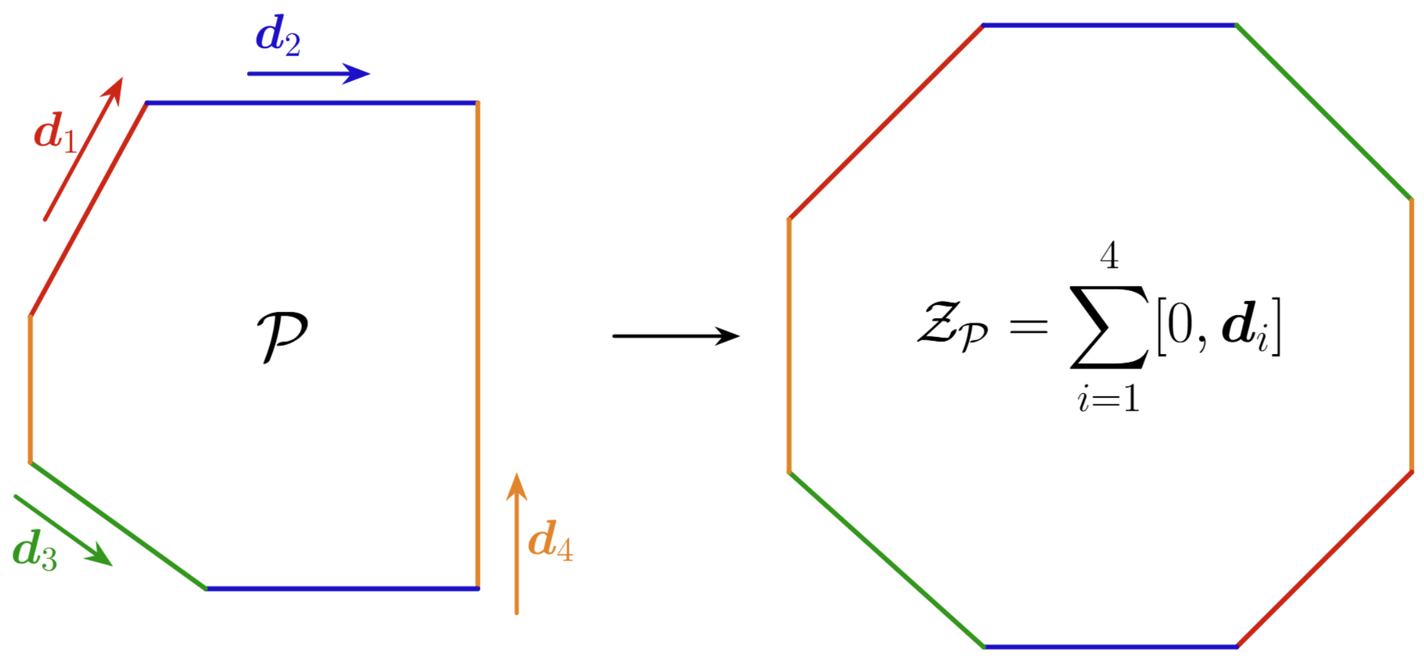
Upper bound for general polytopes

For an edge $e = [\boldsymbol{u}, \boldsymbol{v}]$ of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, its **edge direction** is

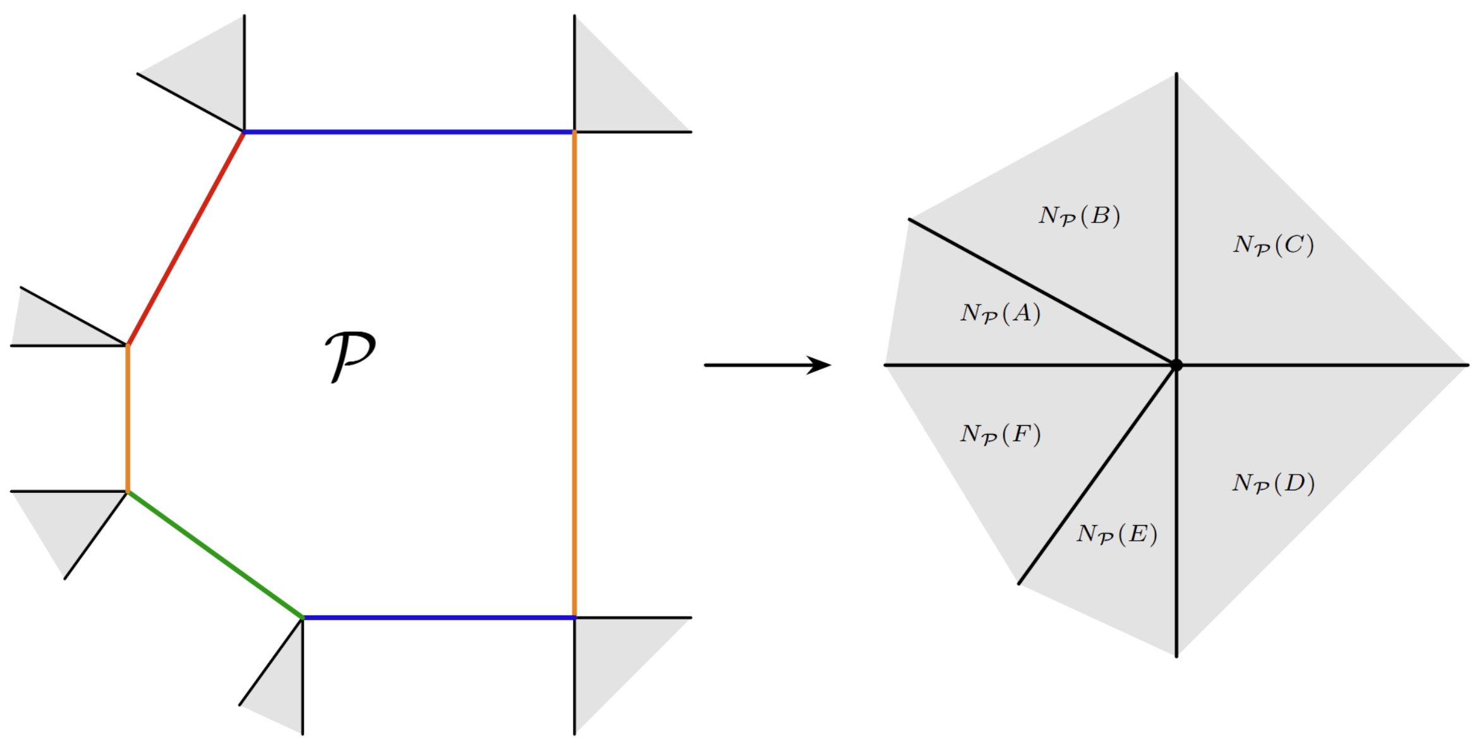
$$\operatorname{dir}(e) = \{\alpha(\boldsymbol{u} - \boldsymbol{v}) : \alpha \neq 0\}.$$

Let $D = \{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_k\} \subseteq \mathbb{R}^n$ cover all edge directions of $\mathcal{P}$:

- Define **Edge-direction zonotope**: $\mathcal{Z}_{\mathcal{P}} = [\boldsymbol{0}, \boldsymbol{d}_1] + \cdots + [\boldsymbol{0}, \boldsymbol{d}_k]$.



- **Upper Bound**: $\sum_{i=0}^{\ell} \binom{k}{i}$ ($\mathcal{Z}_{\mathcal{P}}$ refines the normal fan)



Upper bound for some special polytopes

- If $\mathcal{P}$ is a matroid base polytope, then the upper bound is $\sum_{i=0}^{\ell} \binom{\binom{|E|}{2}}{i}$.
- If $\mathcal{P}$ is a polymatroid, then the upper bound is $\sum_{i=0}^{\ell} \binom{\binom{|E|}{2} + |E|}{i}$.
- If $\mathcal{P} = \mathcal{Z} \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i,\ i = 1, \ldots, m\}$, then the upper bound is $O\left(\binom{2k^{m+1}}{\ell}\right)$.

Lattice polytopes

Integral assumption

- $\mathcal{P}$ is a **lattice polytope**, meaning that all vertices of $\mathcal{P}$ are integral.
- We assume that $\mathcal{P} \subseteq [0, K]^n$ for some $K \in \mathbb{Z}_{\geq 0}$
- We also assume that the objective vectors satisfy $\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell \in \mathbb{Z}^n$.
- Carstensen (1983) studied a related setting with $K = 1$ and $\ell = 1,2$.

Upper bound for lattice polytopes

Given integer vectors $\boldsymbol{c}'_0, \boldsymbol{c}'_1, \ldots, \boldsymbol{c}'_\ell \in \mathbb{Z}^n$ such that

- $\boldsymbol{c}'_1, \ldots, \boldsymbol{c}'_\ell$ are linearly independent,
- $\boldsymbol{c}'_0 + \operatorname{span}\{\boldsymbol{c}'_1, \ldots, \boldsymbol{c}'_\ell\} = \boldsymbol{c}_0 + \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell\}$.

Then the number of critical regions of φ over $\mathcal{P}$ is at most

$$\min\left\{\prod_{i=1}^{\ell}(K\|\boldsymbol{c}'_i\|_1 + 1),\ \min_{j=1,\ldots,\ell}\left[2\prod_{i=0,\, i\neq j}^{\ell}(K\|\boldsymbol{c}'_i\|_1 + 1)\right]\right\}$$

Lattice and successive minima

- We denote by $B_1(\boldsymbol{0}, r)$ the ℓ_1-ball of radius $r > 0$ centered at the origin.
- Define $\mathcal{L} := \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell\} \cap \mathbb{Z}^n$. Then $\mathcal{L}$ is a lattice of rank ℓ.
- For the lattice $\mathcal{L} \subseteq \mathbb{R}^n$ of rank ℓ, define the i-th **successive minimum** by

$$\lambda_i(\mathcal{L}) := \inf\{r > 0 : \dim(\operatorname{span}(\mathcal{L} \cap B_1(\boldsymbol{0}, r))) \geq i\}, \qquad i = 1, \ldots, \ell.$$

Then the number of critical regions of φ over $\mathcal{P}$ is at most

$$\textstyle\prod_{i=1}^{\ell}(K\lambda_i(\mathcal{L}) + 1),$$

Moreover, the values $\{\lambda_i(\mathcal{L})\}_{i=1}^{\ell}$ can be computed in polynomial time when ℓ is fixed.

Conclusion

Our results show that, for fixed ℓ, parametric linear optimization becomes tractable for broad structured classes, including polytopes with few edge directions and lattice polytopes with short objective vectors. In particular, these classes admit polynomially many critical regions and hence polynomial-time parametric algorithms. This identifies large and natural tractable families in the parameter regime, even though the problem is intractable in general.