A Time-Indexed Integer Programming Formulation for the Minimum Broadcast Time Problem

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1. Problem and Contributions

Minimum Broadcast Time. The problem asks how quickly one source vertex can spread a message to every vertex in a network under the telephone model. The network is modeled as a connected graph $G = (V, E)$ with source $s \in V$. The minimum broadcast time is

$$\tau(G, s) = \min\{T : \text{every vertex has received the message by round } T\}.$$

Telephone model.

- Time is divided into discrete rounds.
- In each round, an informed vertex may call at most one neighbor.
- A vertex can transmit only after it has already received the message.

Context.

- **MBT is NP-complete on general graphs[1].**
- Exact results are known for special graph classes such as complete graphs, trees, cycles, and wheels.
- Existing approaches include graph-specific formulas, broadcast-tree heuristics, and time-indexed integer programming models [2,3].

Contributions.

- We develop **Model A**, a time-indexed integer programming formulation for the minimum broadcast time problem.
- We prove that Model A has a tighter LP relaxation than the classical formulation of de Sousa et al. [2].
- We connect Model A to matching-polytope structure, enabling potential blossom cut strengthening.
- Computational experiments show tighter root gaps and faster certification across benchmark graph families.

Applications.

- Peer-to-peer communication networks.
- Sensor and monitoring networks.
- Satellite constellations.

2. The de Sousa et al. Model

For arcs (i, j) induced by edges, define

$$x_{ij}^t = \begin{cases} 1, & \text{if } i \text{ transmits to } j \text{ in round } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \sum_{j \in N(i)} x_{ij}^1 = \begin{cases} 1 & i = s, \\ 0 & i \neq s, \end{cases} && i \in V, \\
& \sum_{t=1}^{T} \sum_{j \in N(i)} x_{ji}^t = 1 && i \in V \setminus \{s\}, \\
& \sum_{j \in N(i)} x_{ij}^t \leq 1 && i \in V,\ t = 2, \ldots, T, \\
& x_{ij}^t \leq \sum_{\tau=1}^{t-1} \sum_{k \in N(i) \setminus \{j\}} x_{ki}^\tau && (i, j) \in A,\ i \neq s,\ t \geq 2, \\
& \sum_{t=1}^{T} t\, x_{ij}^t \leq z && (i, j) \in A, \\
& x_{ij}^t \in \{0, 1\}, \quad z \in \mathbb{Z}_+.
\end{aligned}$$

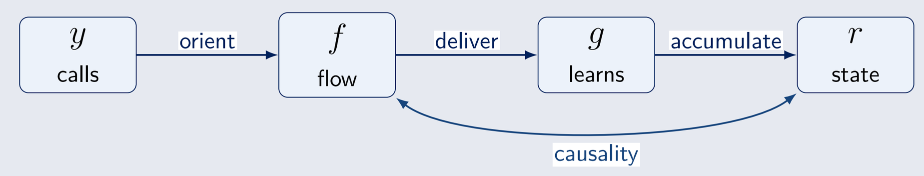
3. Model A

Let

$$A = \{(i, j), (j, i) : \{i, j\} \in E\}.$$

Four-layer decomposition

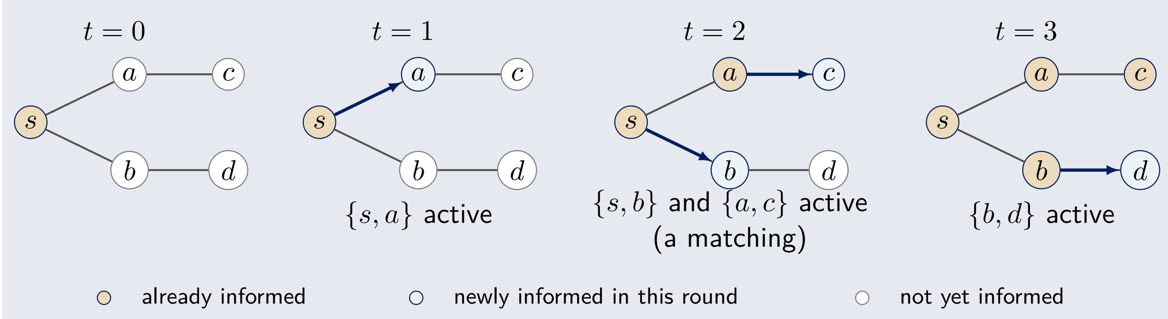
Layer	Variable	Meaning
capacity	$y_{tij} \in \{0,1\}$	$\{i,j\}$ (undirected) used
flow	$f_{tij} \in \{0,1\}$	$i \to j$ transmission
learning	$g_{ti} \in \{0,1\}$	i learns at t
inventory	$r_{ti} \in \{0,1\}$	i informed by t



Integer programming formulation

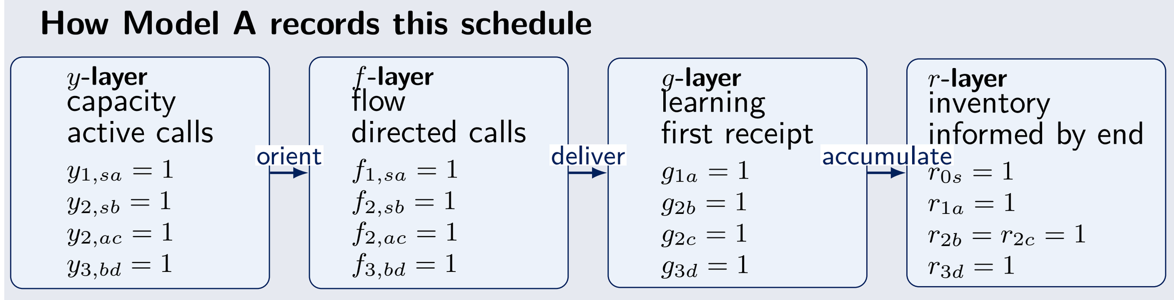
$$\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \sum_{t=1}^{T} t\, g_{ti} \leq z && i \in V \setminus \{s\}, \\
& r_{0s} = 1, \quad r_{0i} = 0 && i \in V \setminus \{s\}, \\
& r_{ti} = r_{t-1,i} + g_{ti} && i \in V \setminus \{s\},\ t = 1, \ldots, T, \\
& r_{ts} = 1 && t = 0, \ldots, T, \\
& \sum_{t=1}^{T} g_{ti} = 1 && i \in V \setminus \{s\}, \\
& g_{tj} = \sum_{i \in N(j)} f_{tij} && j \in V \setminus \{s\},\ t = 1, \ldots, T, \\
& \sum_{j \in N(i)} f_{tij} \leq r_{t-1,i} && i \in V,\ t = 1, \ldots, T, \\
& \sum_{j \in N(i)} y_{tij} \leq 1 && i \in V,\ t = 1, \ldots, T, \\
& f_{tij} + f_{tji} = y_{tij} && \{i, j\} \in E,\ t = 1, \ldots, T, \\
& y, f, g, r \in \{0, 1\}, \quad z \in \mathbb{Z}_+.
\end{aligned}$$

Example broadcast schedule



A broadcast schedule is a sequence of round-by-round transmissions. In each round, active calls must form a matching, and only previously informed vertices may transmit.

How Model A records this schedule



Interpretation. The de Sousa variable x_{ij}^t mixes several roles at once: whether a communication link is used, which direction information travels, when a node first receives the message, and whether a node is already informed. Model A separates these roles into y, f, g, and r, allowing each layer to enforce tighter structural constraints.

4. Theoretical Results

1. Exactness. Every valid broadcast schedule of length T induces a feasible integer solution with the same completion time. Conversely, every feasible integer solution defines a valid telephone-model schedule. For each round t,

$$M_t = \{\{i, j\} \in E : y_{tij} = 1\}$$

is a matching, and f orients each active call. Hence

$$z^\star = \tau(G, s).$$

2. Projection dominance. Let P_A^{LP} be the LP relaxation of Model A and let $\widetilde{P}_{DS}^{LP}$ be the normalized de Sousa relaxation. Under

$$\pi(f, g, r, z) = (x, z), \qquad x_{ij}^t := f_{tij},$$

we have

$$\boxed{\pi(P_A^{LP}) \subseteq \widetilde{P}_{DS}^{LP}}, \qquad z_A^{LP} \geq \tilde{z}_{DS}^{LP}.$$

3. Node-wise dominance. For any consistent partial fixing α of shared transmission variables,

$$\boxed{\pi(P_A^{LP}(\alpha)) \subseteq \widetilde{P}_{DS}^{LP}(\alpha)}, \qquad \min z_A(\alpha) \geq \min z_{DS}(\alpha).$$

This is a bound statement, not a universal runtime theorem.

4. Matching-polytope layer. For each round t,

$$y_t \in \operatorname{conv}\{\chi^M : M \text{ is a matching in } G\}.$$

Model A includes

$$\sum_{j \in N(i)} y_{tij} \leq 1, \qquad y_{tij} \geq 0,$$

and permits odd-set strengthening:

$$\boxed{\sum_{e \in E(U)} y_{t,e} \leq \left\lfloor \frac{|U|}{2} \right\rfloor, \qquad U \subseteq V,\ |U| \text{ odd.}}$$

5. Projection Proof Sketch

Under $x_{ij}^t = f_{tij}$, Model A implies the main DS constraints:

reception: $$\sum_{t=1}^{T} \sum_{j \in N(i)} x_{ji}^t = \sum_{t=1}^{T} g_{ti} = 1,$$

one-peer: $$\sum_{j \in N(i)} x_{ij}^t = \sum_{j \in N(i)} f_{tij} \leq r_{t-1,i} \leq 1,$$

causality: $$x_{ij}^t = f_{tij} \leq r_{t-1,i} = \sum_{\tau=1}^{t-1} \sum_{k \in N(i)} x_{ki}^\tau,$$

makespan: $$\sum_{t=1}^{T} t x_{ij}^t \leq \sum_{t=1}^{T} t g_{tj} \leq z.$$

6. Main Computational Results

Benchmark protocol. Families: cube-connected cycles, De Bruijn, hypercubes, shuffle-exchange, Harary, random geometric, and Watts–Strogatz small-world graphs. Horizon: $T = |V| - 1$; relative MIP gap tolerance 10^{-4}. Main benchmark: no user-defined cuts and no custom branching priorities.

Family	$\lvert V\rvert$	$\lvert E\rvert$	Inst.	DS gap (%)	A gap (%)	DS T (s)	A T (s)	Frac. Opt. (DS/A)
CCC	8	8	7	50.00	0.00	0.003	0.007	1.00/1.00
	24	36	7	65.97	0.00	0.198	0.066	1.00/1.00
	64	96	7	68.52	5.56	3.49	0.705	1.00/1.00
	160	240	7	67.68	3.03	53.09	11.85	1.00/1.00
	384	576	7	98.86	0.00	1229.89	478.67	0.00/1.00
De Bruijn	8	13	7	59.64	10.12	0.010	0.009	1.00/1.00
	16	29	7	58.93	11.31	0.075	0.034	1.00/1.00
	32	61	7	59.69	11.97	0.613	0.146	1.00/1.00
	64	125	7	60.71	10.71	6.23	1.52	1.00/1.00
	128	253	7	61.60	7.50	60.14	15.66	1.00/1.00
RGG	20	25–32	3	7.54	5.48	0.111	0.056	1.00/1.00
	40	111–126	3	55.80	10.66	4.00	0.402	1.00/1.00
	60	244–301	3	57.88	9.33	426.16	2.33	0.67/1.00
	80	485–518	3	74.37	14.29	1203.80	6.97	0.00/1.00
	100	735–778	3	83.99	4.04	918.78	9.82	1.00/1.00
Harary	10	15	7	70.83	12.50	0.018	0.012	1.00/1.00
	20	30	7	66.88	4.17	0.081	0.041	1.00/1.00
	40	60	7	66.73	2.27	0.697	0.128	1.00/1.00
	80	120	7	66.69	1.19	6.21	0.731	1.00/1.00
	160	240	7	66.67	0.61	53.05	3.60	1.00/1.00
Hypercube	8	12	7	66.67	0.00	0.008	0.006	1.00/1.00
	16	32	7	75.00	0.00	0.082	0.031	1.00/1.00
	32	80	7	80.00	0.00	1.30	0.165	1.00/1.00
	64	192	7	83.33	0.00	16.15	2.26	1.00/1.00
	128	448	7	85.71	0.00	208.31	37.72	1.00/1.00
Shuffle	8	10	7	0.00	0.00	0.005	0.006	1.00/1.00
	16	21	7	0.00	0.00	0.049	0.066	1.00/1.00
	32	46	7	6.12	4.59	0.233	0.135	1.00/1.00
	64	93	7	6.75	4.79	2.42	0.633	1.00/1.00
	128	190	7	9.44	5.12	26.48	6.28	1.00/1.00
Small-world	10	30	7	88.14	30.56	0.142	0.030	1.00/1.00
	20	60	7	85.45	22.45	0.534	0.116	1.00/1.00
	40	120	7	83.55	12.36	3.87	0.494	1.00/1.00
	80	240	7	82.29	10.24	32.52	4.36	1.00/1.00
	160	480	7	81.23	9.62	425.29	81.92	1.00/1.00

CCC = cube-connected cycles; RGG = random geometric graphs. Root gap is $\frac{z^{IP} - z^{LP}}{z^{IP}}$. Times are average MIP solve times in seconds.

Interpretation. Across all seven graph families, Model A gives substantially tighter root LP gaps than DS. Runtime gains are especially visible on random geometric, Harary, hypercube, small-world, and large cube-connected-cycle instances. This supports the main claim: separating propagation structure from communication capacity and imposing tighter constraints significantly improves MBT certification.

7. Minor Diagnostic: Blossoms

Pentagon-bouquet stress test. Construct k disjoint induced 5-cycles, each connected to source s by one bridge:

$$|V| = 1 + 5k, \qquad |E| = 6k, \qquad T = |V| - 1.$$

For each pentagon C_5,

$$\sum_{e \in E(C_5)} y_{t,e} \leq 2.$$

k	Variant	Time	Nodes	Bd.	Gap	Cuts
8	Model A	1.83	1617	11	0.0%	0
8	A+blossoms	25.91	51	11	0.0%	10
10	Model A	9.10	3537	13	0.0%	0
10	A+blossoms	1313.23	2122	11	15.4%	58
12	Model A	12.28	2662	15	0.0%	0
12	A+blossoms	1232.82	889	11	26.7%	159

Takeaway. Blossom cuts can reduce node counts, but unrestricted exact separation can dominate runtime.

8. Conclusion

- **Model A is stronger.** Separating calls, flow, learning, and informedness gives a tighter formulation than that of de Sousa et al. [2].
- **The LP bound improves before adding cuts.** The f, g, r propagation layers already dominate the normalized de Sousa relaxation.
- **Blossom cuts help selectively.** Odd-set cuts can reduce search, but exact separation may cost more than it saves.
- **Next directions.** Improve propagation constraints and extend the model to multi-source, multi-message, and weighted variants.

Main message. A stronger formulation gives stronger bounds, and stronger bounds materially improve MBT certification.

9. Selected References

1. Slater, Cockayne, and Hedetniemi. *Information dissemination in trees.* SIAM J. Comput., 10(4):692–701, 1981.
2. de Sousa et al. *Heuristics for the Minimum Broadcast Time.* Electron. Notes Discrete Math., 69:165–172, 2018.
3. Ivanova, Haugland, and Tvedt. *Strong bounds and exact solutions to the minimum broadcast time problem.* Int. Trans. Oper. Res., 32:314–352, 2025.

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