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Isotonic Optimization with Fixed Costs

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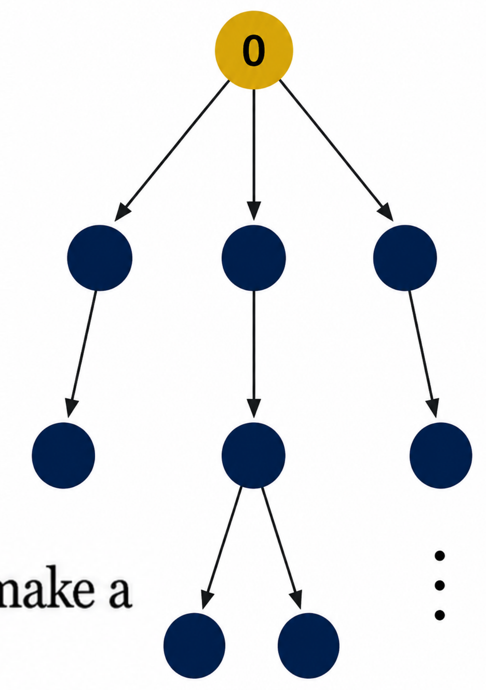
1 Model: Arborescence Formulation

$$\min_{(x,y)\in R^n\times\{0,1\}^n} \sum_{i\in[n]} K_i y_i + \sum_{i\in[n]} f_i(x_i)$$
$$\text{s.t.} \quad (x_{p(i)} - x_i)(1-y_i) = 0, \quad \forall i \in [n],$$
$$x_{p(i)} \le x_i, \quad \forall i \in [n].$$



- Each node may either inherit its parent state or make a strict monotone jump and pay a fixed cost K_i.
- The model generalizes isotonic optimization from paths to arborescences and incorporates fixed costs.
- Core challenge: convex costs plus fixed-charge jumps create a mixed integer nonlinearity structure.

2 Motivating Examples

The framework unifies several monotone decision problems.

(A) Reduced isotonic regression (path case)

$$\min_{x\in\mathbb{R}^n,\, z\in\{0,1\}^{n-1}} \sum_{i\in[n]} K_i y_i + \sum_{i\in[n]} f_i(x_i)$$
$$\text{s.t.} \quad (x_i - x_{i-1})(1-y_i) = 0, \quad \forall i \in [n]\backslash\{1\}$$
$$x_1 \le x_2 \le \cdots \le x_n,$$

(B) Closed-loop stochastic lot sizing (arborescence case)

$$\min_{x\in\mathbb{R}^n,\, z\in\{0,1\}^{n-1}} \sum_{i\in[n]} K_i y_i + \sum_{i\in[n]} f_i(x_i)$$
$$\text{s.t.} \quad (x_i - x_{i-1})(1-y_i) = 0, \quad \forall i \in [n]\backslash\{1\},$$
$$x_1 \le x_2 \le \cdots \le x_n,$$

(C) Other applications include markdown pricing and related monotone control models.

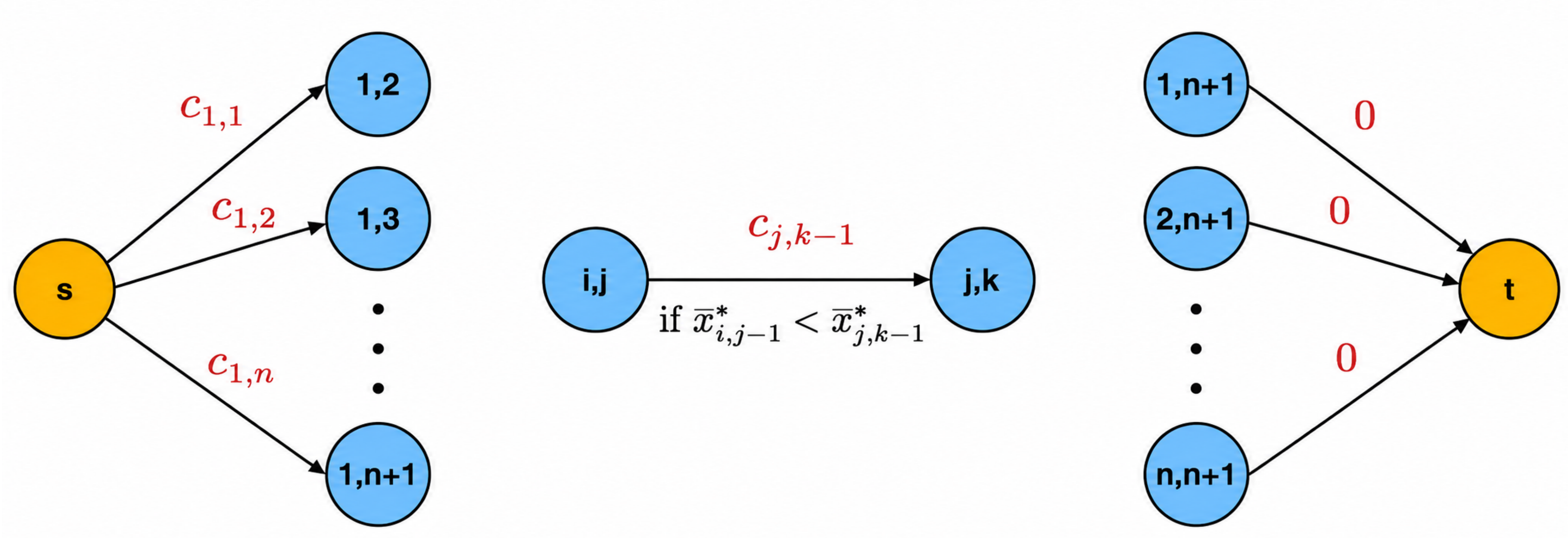
3 Path Case: DP Algorithm and Its Acceleration

Key proposition
An optimal path solution partitions the index set into contiguous blocks. The value within each block equals its unconstrained block minimizer, and the block values must increase across consecutive blocks.

$\overline{x}_{i,j}$ is the block minimizer and $c_{i,j}$ is the block cost.

State $V(j,k)$: best cost for the first $k-1$ indices when the last block is $B(j,k-1)$.

$$V(j,k) = \begin{cases} c_{1,k-1}, & \text{if } j=1,\ 1\le j<k\le n+1,\\ c_{j,k-1} + \min\limits_{\substack{i:1\le i<j,\\ \overline{x}^*_{i,j-1} < \overline{x}^*_{j,k-1}}} V(i,j), & \text{if } 2\le j<k\le n+1. \end{cases}$$



Acceleration

- Efficient implementation computes the path DP in $O(n^2 \log n)$ time.

4 DP for Arborescence Cases with Piecewise Linear Costs

When each f_i is piecewise linear convex, only kink points need to be considered.

$V(i,x)$ = optimal cost of the subtree $T(i)$ given $x_{p(i)} = x$.

Let $K = \{-\infty\} \cup \{\text{kink points of } f_i\}$ be the ordered state set.

$$V_{\rightarrow}(i,k) = f_i(k) + \sum_{j\in C(i)} V(j,k_j).$$
$$V_{\uparrow}(i,k) = \min_{k'\in K: k'>k} \Big\{ K_i + f_i(k') + \sum_{j\in C(i)} V(j,k') \Big\}.$$
$$V(i,k) = \begin{cases} V_{\uparrow}(1,-\infty), & i=1,\ k=-\infty,\\ \min\{V_{\rightarrow}(i,k), V_{\uparrow}(i,k)\}, & \forall i\in[n]\setminus\{1\},\ k\in K. \end{cases}$$

- The state space collapses to the ordered kink set.
- Scanning kink points yields an efficient DP implementation.
- If the total number of kinks is $M = \sum_i m_i$, the runtime is $O((n + \log M)M)$, which is $O(n^2)$ when $m_i = O(1)$.

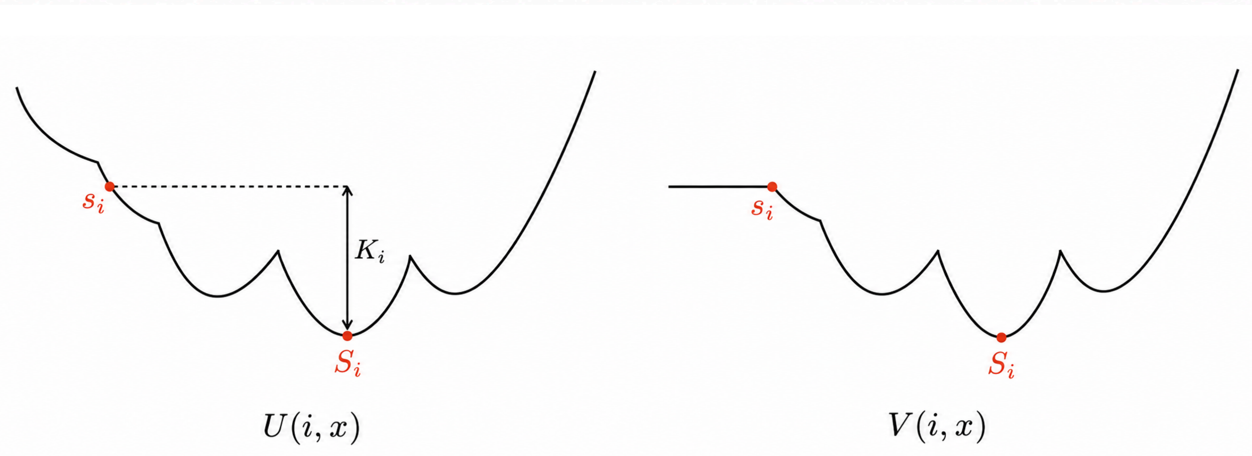
★ The closed-loop SLS model is a special case because each cost has one kink point.

5 General Convex Costs: Bounding the Number of Convex Pieces

(A) Dominating Fixed Costs in Parent Nodes

Assumption: $K_i \ge \sum_{j\in C(i)} K_j, \quad \forall i.$

- The value-to-go function $V(i,x)$ is piecewise convex and K_i-convex.
- Each round introduces at most one new convex piece.



(B) Special Graph Structures

- **Lemma:** if g has L convex pieces and R quasi-convex pieces, then $h(x) = \min\limits_{x'\ge x} \{K1\{x'>x\} + g(x')\}$ has at most $L+R$ convex pieces.
- Balanced arborescences (height $O(\log n)$) and trees with only $O(1)$ branching rounds admit polynomially many convex pieces.