# SDP Approach to Quadratic $k$-Vertex-Disjoint Paths Problem

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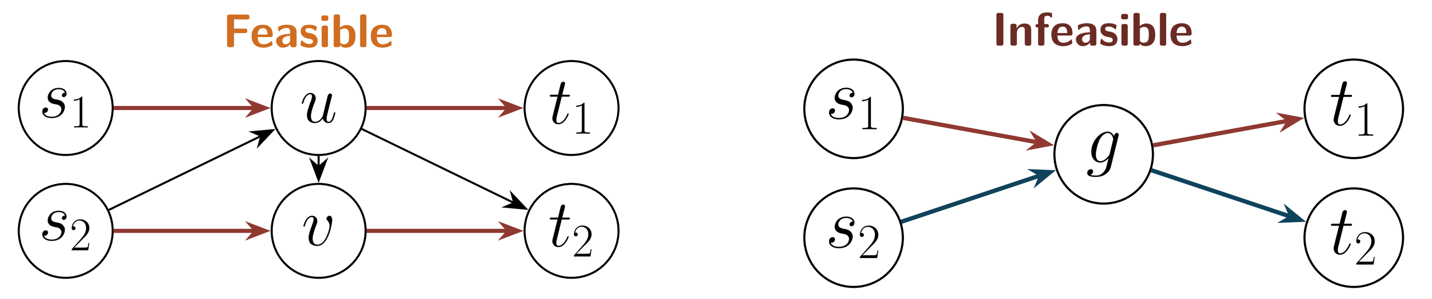
## 1. Introduction

The **$k$-vertex-disjoint paths problem ($k$-VDP)** asks whether each terminal pair $(s_i, t_i)$ in a directed graph $G = (V, E)$ can be connected by paths that do not share vertices. The problem is **NP-hard** [1].

We study **Quadratic $k$-VDP (Q-$k$-VDP)**, where feasible disjoint paths minimize a **nonconvex quadratic** cost over arcs and arc-pair interactions.




*Vertex-disjoint paths.* *Shared bottleneck vertex.*

## 2. Problem Formulation

**Step I** **Copy the graph for each path.** For each $i = 1, \ldots, k$, define $G_i = (N_i, E_i)$ by

$$N_i = \{v^i : v \in N\}, \quad E_i = \{(u^i, v^i) : (u, v) \in E\}.$$

**Step II** **Build the disjoint union.** Let $kG = \bigsqcup_{i=1}^k G_i$, with $m = k|N|$ vertices and $n = k|E|$ arcs. Selecting one $s_i \to t_i$ path in each copy handles routing; vertex-disjointness allows **at most one copy** of each original vertex to be used.

**Step III** **Formulate the resulting BQP.** Let $A$ be the vertex–arc incidence matrix of $kG$, $b$ the supply–demand vector, and $x \in \{0,1\}^n$ the arc-selection vector:

$$\begin{aligned} &\min_{x\in\{0,1\}^n} x^\top Q x && \textit{(Q symmetric \& indefinite)}\\ &\text{s.t. } Ax = b, && \textit{(flow conservation)}\\ &\left.\begin{aligned}&\textstyle\sum_{i=1}^k \sum_{e\in\delta_+(v^i)} x_e \le 1, \quad \forall v \in N,\\ &\textstyle\sum_{i=1}^k \sum_{e\in\delta_-(v^i)} x_e \le 1, \quad \forall v \in N,\end{aligned}\right\} && \textit{(vertex-disjointness)}\\ &\textstyle\sum_{e\in E_i(S)} x_e \le |S| - 1, \ \forall i, \ \forall S \subseteq N_i, |S| \ge 2. && \textit{(subtour elimination)} \end{aligned}$$

## 3. Graph Reduction

The disjoint-union graph $kG$ can yield a **high-dimensional** BQP. For planar graphs and fixed $k$, $k$-VDP is polynomially solvable [2]; we use this to identify arcs that are **never used** or **must be used**.

### Fixed-Zero and Fixed-One Arcs

**Fixed-zero arc.** An arc $e = (u^i, v^i)$ is called fixed-zero if $x_e = 0$ for every feasible solution. To test this, construct an auxiliary $(k+1)$-VDP instance on $G$ by replacing $(s_i, t_i)$ with $(s_i, u)$ and $(v, t_i)$, leaving all other terminal pairs unchanged.

**Lemma 1** *An arc $e = (u^i, v^i)$ in $kG$ is fixed-zero if and only if the resulting auxiliary $(k+1)$-VDP instance on $G$ is infeasible.*

**Fixed-one arc.** An arc $e = (u^i, v^i)$ is called fixed-one if $x_e = 1$ for every feasible solution. Let $\tilde{G}$ be obtained from $G$ by deleting $(u, v)$ and test the original $k$ terminal pairs on $\tilde{G}$.

**Lemma 2** *An arc $e = (u^i, v^i)$ in $kG$ is fixed-one if and only if the auxiliary $k$-VDP instance on $\tilde{G}$ is infeasible and $(u^j, v^j)$ is fixed-zero for all $j \ne i$.*

## 3.1. Reduced Formulation

Let $E^0$ and $E^1$ be the arcs fixed to zero and one, respectively. Removing them from $kG$ gives the **reduced graph** $kG_R = (N', E')$, with $n' = |E'|$ remaining arcs, incidence matrix $A'$, and $b' = b - \sum_{e\in E^1} A_e$.

The **Reduced BQP (R-BQP)** uses $x' \in \{0,1\}^{n'}$:

$$\min\ (x')^\top Q' x' + (c')^\top x' + \kappa \quad \text{s.t.} \quad A'x' = b',$$

with vertex-disjointness and subtour constraints restricted to $kG_R$.

**Theorem (Equivalence)** *Given $x' \in \{0,1\}^{n'}$, define a binary vector $x$ on $kG$ by $x_{E'} = x', x_{E^1} = \mathbf{1}, x_{E^0} = \mathbf{0}$. Then $x'$ is feasible for the R-BQP if and only if $x$ is feasible for the BQP. Moreover, their objective values coincide.*

## 3.2. Impact of Graph Reduction

We test the reduction on feasible Q-$k$-VDP instances generated from directed grid graphs with $m_v$ vertices. For each configuration $(m_v, k)$, we generate 40 instances.

**Representative statistics:** average arc counts before/after reduction and average reduction time.

| $k$ | $m_v$ | Original Arcs $n$ | Reduced Arcs $n'$ | Time (s) |
|---|---|---|---|---|
| 2 | 20 | 100 | 55 | 0.05 |
| 3 | 40 | 342 | 168 | 0.38 |
| 6 | 60 | 1009 | 180 | 1.66 |
| 8 | 80 | 1832 | 241 | 4.56 |
| 13 | 100 | 3550 | 157 | 10.78 |

**~96% of arcs removed!**

## 4. SDP Relaxation for R-BQP

**Motivation.** R-BQP has a **nonconvex quadratic objective** and **exponentially many subtour constraints**. Dropping subtours yields a **subtour-relaxed R-BQP**, still **NP-hard**; we use an **SDP relaxation** for stronger lower bounds.

Let $\mathcal{C}$ be incompatible arc pairs in $kG_R$ and $Y \in \mathcal{S}_+^{n'+1}$ the lifted variable. Define $\widehat{Q} = \begin{bmatrix} \kappa & \frac{1}{2}(c')^\top \\ \frac{1}{2}c' & Q' \end{bmatrix}$ and $M_A = [-b' \mid A']^\top [-b' \mid A']$.

$$\operatorname{arrow}(Y) := \left(Y_{0,0},\ Y_{1,1} - Y_{0,1},\ \ldots,\ Y_{n',n'} - Y_{0,n'}\right)^\top.$$

### SDP Relaxation

$$\begin{aligned} \min_{Y} \quad & \langle \widehat{Q}, Y\rangle \\ \text{s.t.} \quad & \langle M_A, Y\rangle = 0, \\ & Y_{p,q} = 0, \ \forall (p,q) \in \mathcal{C}, \\ & \operatorname{arrow}(Y) = e_0, \\ & Y \succeq 0. \end{aligned}$$

Thus Slater **fails**: $\langle M_A, Y\rangle = 0$ exposes a proper PSD face.

### Facial Reduction

Let $V \in \mathbb{R}^{(n'+1)\times r}$ be an orthonormal basis for $\ker(M_A)$, $r = \dim\ker(M_A)$. Then $Y = VRV^\top$, $R \in \mathcal{S}_+^r$, parameterizes the face:

$$\begin{aligned} \min_{Y,R} \quad & \langle \widehat{Q}, Y\rangle \\ \text{s.t.} \quad & Y = VRV^\top, \\ & Y_{p,q} = 0, \ \forall (p,q) \in \mathcal{C}, \\ & \operatorname{arrow}(Y) = e_0, \\ & R \succeq 0. \end{aligned}$$

## 5. ADMM Solver and Branch-and-Bound

We solve the facially reduced SDP by tailored alternating direction method of multipliers (ADMM).

**Augmented Lagrangian** ($\beta > 0$, dual $Z$):

$$\mathcal{L}(R, Y, Z) = \langle \widehat{Q}, Y\rangle + \langle Z, Y - VRV^\top\rangle + \tfrac{\beta}{2}\|Y - VRV^\top\|_F^2.$$

**ADMM updates** ($\rho \in (0, \frac{1+\sqrt{5}}{2})$):

$$\begin{aligned} R^{\ell+1} &= \arg\min_{R\succeq 0} \mathcal{L}(R, Y^\ell, Z^\ell), \\ Y^{\ell+1} &= \arg\min_{Y\in\mathcal{F}_Y} \mathcal{L}(R^{\ell+1}, Y, Z^\ell), \\ Z^{\ell+1} &= Z^\ell + \rho\beta(Y^{\ell+1} - VR^{\ell+1}V^\top), \end{aligned}$$

where $\mathcal{F}_Y = \{Y \in \mathcal{S}^{n'+1} \mid Y_{p,q} = 0\ \forall (p,q) \in \mathcal{C},\ \operatorname{arrow}(Y) = e_0\}$. Let $\widetilde{Y}, \widetilde{Z}$ be ADMM iterates and $\mathcal{X}$ the subtour-relaxed R-BQPfeasible set.

We embed ADMM bounds in branch-and-bound (B&B) for exact solution.

### Lower Bound

Project the dual iterate:

$$Z_{\text{proj}} = P_{\mathcal{F}_Z}(\widetilde{Z}), \quad \text{LB} = g(Z_{\text{proj}}),$$

where $\mathcal{F}_Z = \{Z \mid -V^\top Z V \succeq 0\}$ and $g(Z) = \min_{Y\in\mathcal{F}_Y}\langle \widehat{Q} + Z, Y\rangle$.

### Upper Bound

Extract diagonal weights and round:

$$w_p = (\widetilde{Y})_{pp}, \quad p = 1, \ldots, n',$$
$$x^\star = \arg\max\{w^\top x : x \in \mathcal{X}\},$$
$$\text{UB} = (x^\star)^\top Q' x^\star + (c')^\top x^\star + \kappa.$$


## 6. Computational Experiments

We compare our SDP-based method with **Gurobi 11** on solving the subtour-relaxed R-BQP. The full dataset contains 1,160 instances on directed grid graphs. Instances are grouped into five arc bins by $|E'|$; 80 random instances per bin; 1 h time limit.

**Metrics reported:** average relative optimality gap, number of instances solved to optimality, and average runtime.

| Arc bin | Avg. Rel. Gap | | Solved / 80 | | Avg. Time (s) | |
|---|---|---|---|---|---|---|
| | SDP | Gurobi | SDP | Gurobi | SDP | Gurobi |
| $[1, 200)$ | **0.000** | 0.002 | **80** | 79 | **7.16** | 79.85 |
| $[200, 400)$ | **0.069** | 0.373 | **64** | 41 | **945.31** | 2137.35 |
| $[400, 600)$ | **0.335** | 0.992 | **40** | 6 | **2420.18** | 3387.50 |
| $[600, 800)$ | **0.870** | 1.879 | 2 | 2 | 3553.62 | 3549.29 |
| $\ge 800$ | **0.972** | 2.272 | **1** | 0 | 3653.39 | 3600.18 |

*Note:* Times may slightly exceed 3600s because termination is checked after node evaluation.

**Smaller gaps · 1.46× solved · faster on small/medium bins**

## 7. References

[1] Richard M. Karp. On the computational complexity of combinatorial problems. *Networks*, 5(1):45–68, 1975.

[2] Alexander Schrijver. Finding $k$ disjoint paths in a directed planar graph. *SIAM Journal on Computing*, 23(4):780–788, 1994.

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