On the Convexification of a Class of Mixed-Integer Conic Sets

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Convexification of Mixed-Integer Conic Sets

The SOC reformulation belongs to the family

$$Z := \{(x,y) \in X \times \mathbb{R}^m : \|Ax + By + d\|_2 \le f(x)\},$$

where X is a compact mixed-integer set.

The convexification theorem. Under some regularity conditions, the convex hull of Z is given by

$$\mathrm{conv}(Z) = \left\{(x,y) \in \mathrm{conv}(X) \times \mathbb{R}^m : \|Ax + By + d\|_2 \le \widehat{f}(x)\right\},$$

where $\widehat{f}$ is the concave envelope of f over $\mathrm{conv}(X)$, and the convex hull is closed.

Interpretation.

- The SOC left-hand side stays unchanged.
- Convexification occurs entirely through the replacement $f \mapsto \widehat{f}$.
- The same template also extends to any norm.

Positioning relative to prior work. The convexification result follows from existing conic mixed-binary convex hull theory in [3] after a proper reformulation through an affine bijection in an extended space.

Application to MIQCP

Consider a mixed-binary quadratic constraint with a positive definite continuous block:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy}^\top & Q_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a_x^\top x + a_y^\top y \le \text{Constant}, \qquad Q_{yy} \succ 0. \tag{1}$$

Completing the square in y gives an equivalent second-order-cone form

$$\|A_x x + A_y y + b\|_2 \le \sqrt{x^\top Q x + a^\top x + r} =: \sqrt{q(x)}. \tag{2}$$

Thus characterizing the convex hull of the region defined by the quadratic constraint (1) reduces to characterizing the concave envelope of $\sqrt{q(x)}$ over $\mathrm{conv}(X)$.

A **tractable relaxation** approximates $\sqrt{q}$ by $\sqrt{\widehat{q}}$, where $\widehat{q}$ is the concave envelope of q over $\mathrm{conv}(X)$. The task of approximating $\widehat{q}$ can be handled automatically by off-the-shelf solvers, but it still matters how we **guide the solver toward approximating $\widehat{q}$ through reformulation**.

For the most interesting case, where x is **binary**, we propose two reformulation methods, **BQP-based** and **SDP-based**, and compare them in the computational study.

BQP-based Reformulation

A BQP-based tractable surrogate is to expose the concave-envelope problem for the quadratic $q(x)$ itself:

$$\|A_x x + A_y y + b\|_2 \le \eta, \qquad \eta^2 \le \tau, \qquad \tau \le q(x). \tag{3}$$

The constraint $\tau \le q(x)$ in (3) guides the solver toward approximating $\widehat{q}$ in a lifted space using classical cutting-plane methods, including BQP cuts and RLT cuts.

SDP-based Conic Reformulation

Consider the problem

$$\max\{p_x^\top x + p_y^\top y : \text{s.t. } (2),\ (x,y) \in X \times Y\},$$

where $X \subseteq \{0,1\}^n$ and $Y \subseteq \mathbb{R}^m$ are compact sets, and the continuous relaxation $\overline{X \times Y}$ is convex with nonempty interior. Define the convex relaxation via a diagonal shift γ as

$$(\mathrm{CR}_\gamma) \quad v(\gamma) = \max \quad p_x^\top x + p_y^\top y$$
$$\text{s.t.} \quad \|A_x x + A_y y + b\|_2 \le \eta, \tag{4a}$$
$$\eta^2 + x^\top(\mathrm{Diag}(\gamma) - Q)x - (\gamma + a)^\top x - r \le 0, \tag{4b}$$
$$(x,y) \in \overline{X \times Y}.$$

The SDP-based reformulation seeks **the tightest convex relaxation obtainable by a diagonal shift**, i.e., it solves

$$(\mathrm{CR}_\gamma^*) \quad v^* = \inf_\gamma \quad v(\gamma) \quad \text{s.t.} \quad \mathrm{Diag}(\gamma) - Q \succeq 0.$$

Instead of solving the nonconvex problem above directly, we solve the following SDP:

$$(\mathrm{SDP}) \quad \max_{X,x,y,\eta} \quad p_x^\top x + p_y^\top y$$
$$\text{s.t.} \quad \|A_x x + A_y y + b\|_2 \le \eta,$$
$$\eta^2 - \langle Q, X\rangle - a^\top x - r \le 0,$$
$$X \succeq xx^\top,$$
$$\mathrm{diag}(X) = x,$$
$$(x,y) \in \overline{X \times Y}.$$

The bound theorem. The optimal value of (SDP) is equal to the value of the best convex relaxation (CR_γ^*). Under a mild dual nondegeneracy condition, the optimal γ^* can be recovered from an optimal dual solution of (SDP).

Observation. The SDP-based reformulation provides tight root relaxations equal to the SDP bound, which can hardly be improved by the solver at the root node.

DRCC-MB-KP Test Bed

The computational study uses distributionally robust chance-constrained mixed-binary knapsack instances. After the mean-covariance chance constraint is converted to a deterministic quadratic constraint using the conic reformulation in [2], we are able to formulate it as an MIQCP, and hence apply the convexification techniques above.

Three tested models.

- CCP: direct MIQCP formulation.
- BQP: BQP-based reformulation (3)
- SDP: SDP-based conic reformulation (4), where γ is obtained by solving (SDP).

Computational Results

(n,m)	Type	Root Gap (%)			Solved			Final Gap (%)			Time (s)			Nodes		
		CCP	BQP	SDP	CCP	BQP	SDP	CCP	BQP	SDP	CCP	BQP	SDP	CCP	BQP	SDP
(25,25)	1	695.9	272.5	**262.3**	4/5	**5/5**	**5/5**	409.7	**0.0**	**0.0**	787.6	1.4	**0.3**	9.7×10^4	6.6×10^2	3.1×10^2
	2	450.2	**113.7**	129.3	3/5	**5/5**	**5/5**	22.7	**0.0**	**0.0**	1479.4	1.4	**0.3**	1.2×10^5	8.8×10^2	4.1×10^2
	3	431.0	**66.3**	120.2	**5/5**	**5/5**	**5/5**	**0.0**	**0.0**	**0.0**	423.3	5.3	**0.5**	8.5×10^4	2.2×10^3	5.9×10^2
	4	303.0	**43.2**	257.9	**5/5**	**5/5**	**5/5**	**0.0**	**0.0**	**0.0**	744.2	1.3	**0.4**	7.9×10^4	8.3×10^2	4.3×10^2
(50,50)	1	250.8	134.4	**49.0**	0/5	4/5	**5/5**	309.2	2.2	**0.0**	3600.2	931.5	**8.5**	3.4×10^5	1.1×10^4	3.4×10^3
	2	427.4	273.3	**79.2**	0/5	2/5	**5/5**	411.6	3.3	**0.0**	3600.1	2562.7	**160.3**	4.0×10^5	6.2×10^4	8.9×10^4
	3	394.2	235.0	**69.2**	0/5	2/5	**5/5**	422.9	3.6	**0.0**	3600.1	2676.3	**61.3**	4.2×10^5	5.7×10^4	3.3×10^4
	4	260.8	109.4	**48.8**	2/5	4/5	**5/5**	14.2	1.0	**0.0**	2196.8	1172.3	**26.2**	2.5×10^5	2.9×10^4	1.4×10^4
(100,100)	1	148.6	85.3	**20.8**	(1) 0/5	0/5	**1/5**	160.2	63.5	**5.5**	2882.1	3600.1	**2891.3**	1.8×10^5	7.4×10^2	4.5×10^5
	2	254.8	187.7	**76.1**	0/5	0/5	0/5	333.3	189.2	**18.7**	3601.7	3600.1	3600.0	2.5×10^5	6.1×10^2	6.1×10^5
	3	254.4	181.2	**71.7**	(3) 0/5	0/5	**1/5**	395.9	198.3	**25.6**	1441.0	3600.4	**3411.5**	9.1×10^4	5.9×10^2	5.5×10^5
	4	226.9	173.7	**40.5**	0/5	0/5	**2/5**	263.1	154.1	**5.0**	3600.6	3600.1	**3012.8**	3.5×10^5	5.9×10^2	5.2×10^5

Each row averages five instances. Numbers in parentheses indicate the number of instances terminated prematurely due to numerical issues. The best root gaps, final gaps, times, and nodes are highlighted in bold. For SDP, the computational time includes the time for solving (SDP).

Takeaways

1. **Exact geometry:** for this conic family, the convex hull is obtained by replacing f with its concave envelope.
2. **Two tractable approximations:** expose the binary quadratic hypograph for BQP/RLT cuts, or compute an SDP-optimal diagonal shift for a stronger MISOCP.
3. **Computational effect:** both reformulations improve computational performance, with the SDP approach delivering the most pronounced gains on the DRCC-MB-KP benchmark.
4. **Drawback of BQP-based reformulation:** the extra lifting variables and abundant cuts can substantially slow down branch-and-bound as the instance size grows, even though they can significantly improve root gaps.

References

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