Sensitivity Analysis for MIPs using Lagrangian dual multipliers and B&B trees

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Background

Consider the following mixed-integer linear program:

$$\begin{aligned} z(\Delta) = \min\,& (c+\Delta_c)^\top x & (P(\Delta)) \\ \text{s.t. }& (A+\Delta_A)x \geq b + \Delta_b, \\ & 0 \leq x \leq 1, \\ & x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}. \end{aligned}$$

Where we parametrized small perturbations of A, b and c by $\Delta = (\Delta_A, \Delta_b, \Delta_c)$. We say $z(\Delta)$ is the value function of $P(\Delta)$ with respect to Δ.

Our goal is to find a valid dual bound $f(\Delta) \leq z(\Delta)$ with good balance between the **strength** of the bound and the **computational effort** required to evaluate it.

LP Sensitivity

Consider the **LP setting**, i.e., when $\mathcal{I} = \emptyset$, and let λ_0^* be the dual maximizer for $P(0)$. We propose the dual function (1), with $[y]^- := \min\{y, 0\}$.

$$\begin{aligned} f_{LP}(\Delta) &:= \min_{0\leq x\leq 1} \sum_{i=1}^n \left((c+\Delta_c)_i - \lambda_0^{*\top}(A+\Delta_A)_i \right) x_i + \lambda_0^{*\top}(b+\Delta_b) \\ &= \sum_{i=1}^n \left[(c+\Delta_c)_i - \lambda_0^{*\top}(A+\Delta_A)_i \right]^- + \lambda_0^{*\top}(b+\Delta_b) \qquad (1) \end{aligned}$$

This comes from evaluating λ_0^* in the bounded Lagrangian relaxation of $P(\Delta)$.

Prop 1: If $P(0)$ is feasible, then $f_{LP}(\Delta)$ is a valid generalized dual, that is

$$f_{LP}(\Delta) \leq z_{LP}(\Delta),\ \forall \Delta \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n.$$

This holds, because if $P(0)$ is feasible, then λ_0^* is well-defined and dual feasible for $P(\Delta)$. Then by weak duality we have that:

$$z_{LP}(\Delta) \geq f_{LP}(\Delta)$$

We have shown that this lower bound has the following property

Theorem 1

Let $P(0)$ and $P(\Delta)$ be feasible. If $F_{A,b} = \{x \in \mathbb{R}^n : Ax \geq b, 0 \leq x \leq 1\}$ is full dimensional with $x^0 \in \text{int}(F_{A,b})$, then for all Δ such that:

$$\forall i \in [m] : [\Delta_b]_i + \sum_{j\in[n]} |[\Delta_A]_{ij}| \leq (Ax^0 - b)_i$$

we have that:

$$z_{LP}(\Delta) - f_{LP}(\Delta) \leq (m+n)\kappa_A \|c\|_1 \|\Delta_b\|_\infty + 2\|\Delta_c\|_1 + (n+1)\kappa_A\|c\|_1\|\Delta_A\|_1.$$

Where $\kappa_A = \max\{\|B^{-1}\|_\infty : B \text{ is an invertible submatrix of } A\}$.

MIP Sensitivity

For the general **MIP setting**, let $\mathcal{T}$ be a pure branch-and-bound tree for the nominal problem $P(0)$. Note that each node v is associated with an LP, and thus admits its own LP value function $z^v(\Delta)$ with the corresponding bounds for v.

Using the approach from Schrage and Wolsey [2], we can define

$$f_{IP}(\Delta) := \min_{v \in \mathbb{L}(\mathcal{T})} f^v_{LP}(\Delta)$$

as a lower bound for $z(\Delta)$, where $\mathbb{L}(\mathcal{T})$ corresponds to the leaves of the tree.

This dual bound can be brittle: a single weak leaf bound $f^v_{LP}(\Delta)$ will dominate the minimum and significantly degrade the resulting bound. To avoid this issue, we propose three methods to strengthen the node bounds.

Tree search

- **Bottom-Up**: Start with $\mathcal{T}$ and try to prune the bottleneck leaf node, until no improvement can be obtained.
- **Top-Down**: Start from the root node and try to add children from $\mathcal{T}$, until no improvement can be obtained.
- **Full-Propagation:** Traverse every node in $\mathcal{T}$ in post-order updating the bounds to the best one possible.

Experimental Setup

First, we solved instances of the CFLP-medium dataset from Distributional MIPLIB [1] using a custom B&B algorithm to recover the search trees. These instances have 200 facilities and 100 customers.

$$\begin{aligned} \min_{x,y}\ & \sum_{c\in C}\sum_{f\in F} w^t_{c,f} x_{c,f} + \sum_{f\in F} w^s_f y_f & & \text{(CFLP)} \\ \text{s.t. } & \sum_{f\in F} x_{c,f} \geq 1, & \forall c \in C & \\ & \sum_{c\in C} d_c x_{c,f} \leq b_f y_f, & \forall f \in F & \\ & \sum_{f\in F} b_f y_f \geq B & & \\ & x_{c,f} \leq y_f, & \forall c \in C,\quad \forall f \in F & \\ & 0 \leq x_{c,f} \leq 1, & \forall c \in C,\quad \forall f \in F & \\ & y_f \in \{0,1\}, & \forall f \in F & \end{aligned}$$

Then, we randomly selected 15 instances depending on the tree size and generated perturbed problems with the following settings:

- **Sparsity level** $S \in \{1\%, 5\%, 10\%\}$ for A and c. For b we only could change one constraint (total capacity).
- **Magnitude change** $\Delta \in \{\pm1\%, \pm5\%, \pm10\%\}$ for A, b and c

Finally, we evaluated 4 different bounds (min leaf and the heuristics) using the tree from the nominal problem.

Computational Results

The results are reported using three configurations for S and Δ. Low=$\{S = 1\%; \Delta_i = \pm1\%\}$. Medium=$\{S = 5\%; \Delta_i = \pm5\%\}$. High=$\{S = 10\%; \Delta_i = \pm10\%\}$. Where i are the parameters not being perturbed.

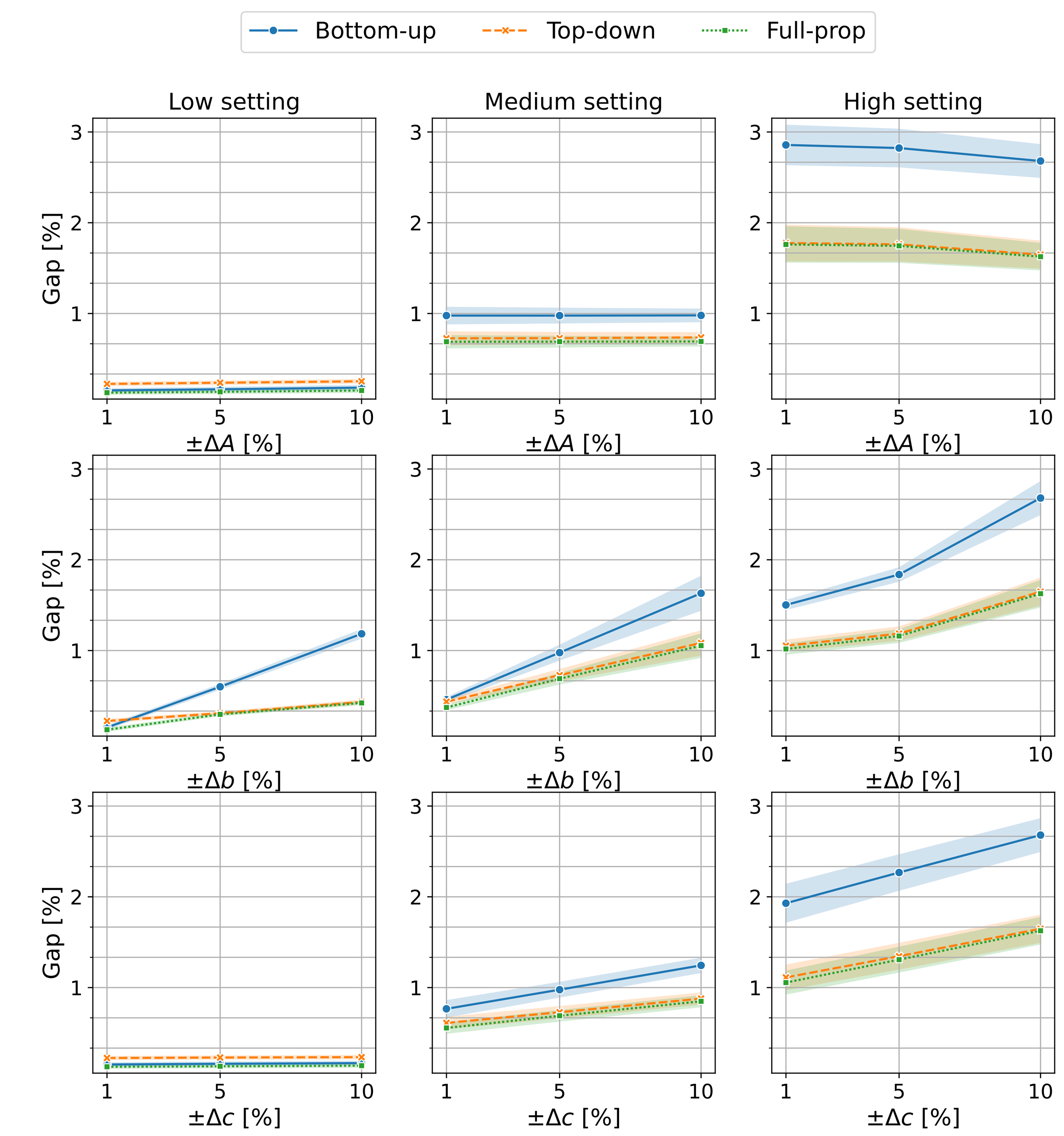

Figure 1. Effects of perturbing A, b and c for three configurations of S and Δ.

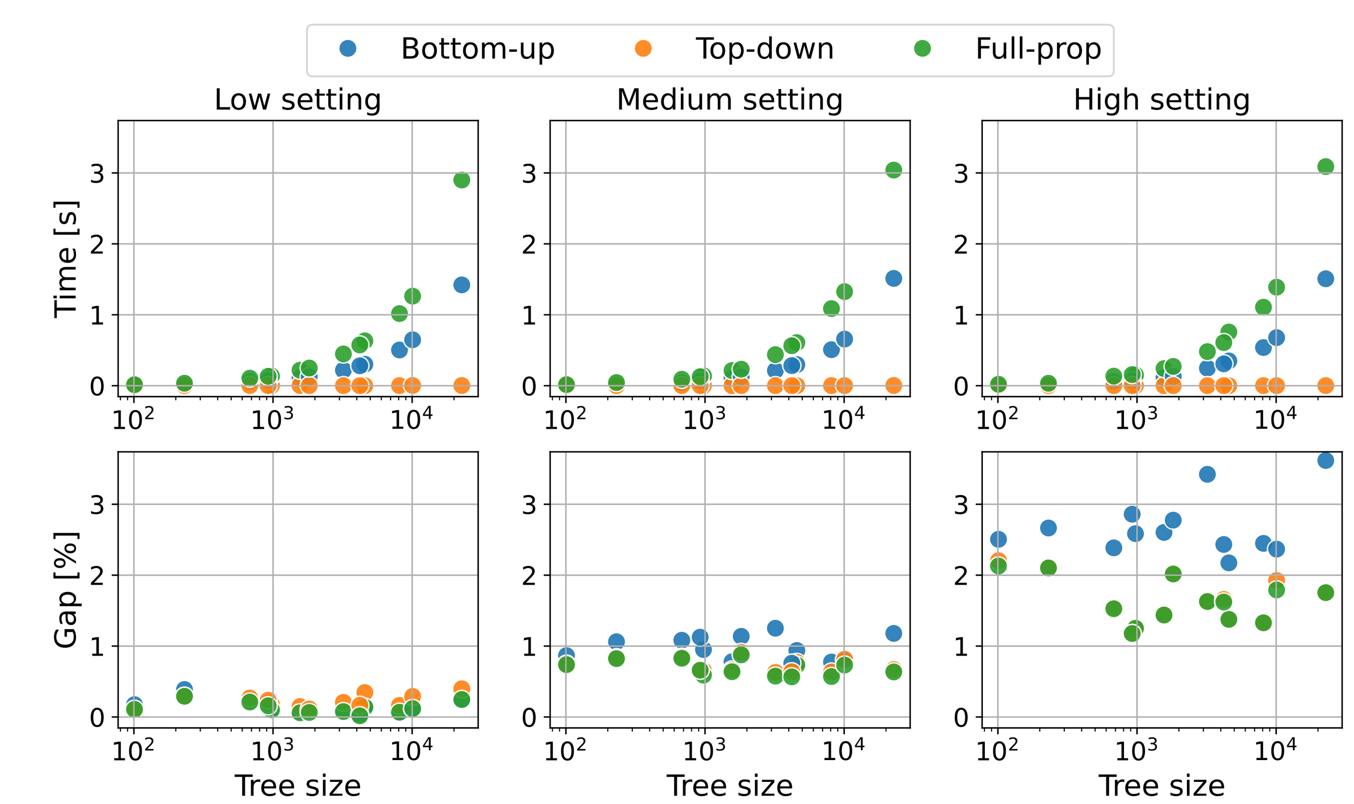

Figure 2. Relation between tree size, the time to evaluate the methods and the gap obtained for three configurations of S and Δ

[1] W. Huang, T. Huang, A. M. Ferber, and B. Dilkina. Distributional MIPLIB: a multi-domain library for advancing ml-guided milp methods, 2024. [2] L. Schrage and L. Wolsey. Sensitivity analysis for branch and bound integer programming, 1985.

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