This research was supported by the Air Force Office of Scientific Research.
Award number: FA9550-23-1-0340

An output sensitive polynomial time algorithm for enumeration of the vertices of the reverse polar set related to simple disjunctions

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The reverse polar set for simple disjunctions ($S^\#$)

Disjunctive set

$$S = \bigcup_{t\in\mathcal{T}} S^t, \qquad S^t = \{x \in \mathbb{R}^n_{\geq 0} : d^t x \geq d^t_0\}$$

Partition variables by coefficient sign

$$\mathcal{J}^- = \{k \in [n] : \forall t \in \mathcal{T}, d^t_k < 0\}$$

$$\mathcal{J}^+ = \{j \in [n] : \exists t \in \mathcal{T}, d^t_k > 0\}$$

$\mathcal{V}$-polyhedral representation of $S^\#$

$$S^\# = \{\alpha \in \mathbb{R}^n : \alpha x \geq 1, \forall x \in S\}$$

$$= \{\alpha \in \mathbb{R}^n : \alpha p \geq 1, \forall p \in \mathcal{P}, \alpha r \geq 0, \forall r \in \mathcal{R}\}$$

$$= \left\{ \begin{array}{c} \alpha \in \mathbb{R}^n : \alpha_j \geq \max\{d^t_j\}, \forall j \in \mathcal{J}^+, \\ \min\left\{\frac{|d^t_k|}{d^t_j}\right\} \alpha_j + \alpha_k \geq 0, \forall (j,k) \in \mathcal{J}^+ \times \mathcal{J}^- \end{array} \right\}$$

Visualizing the disjunctive region

$$S = \{x \in \mathbb{R}^4_{\geq 0} : (x_1 - x_3 - x_4 \geq 1) \vee (2x_2 - 2x_3 - 4x_4 \geq 1)\}$$

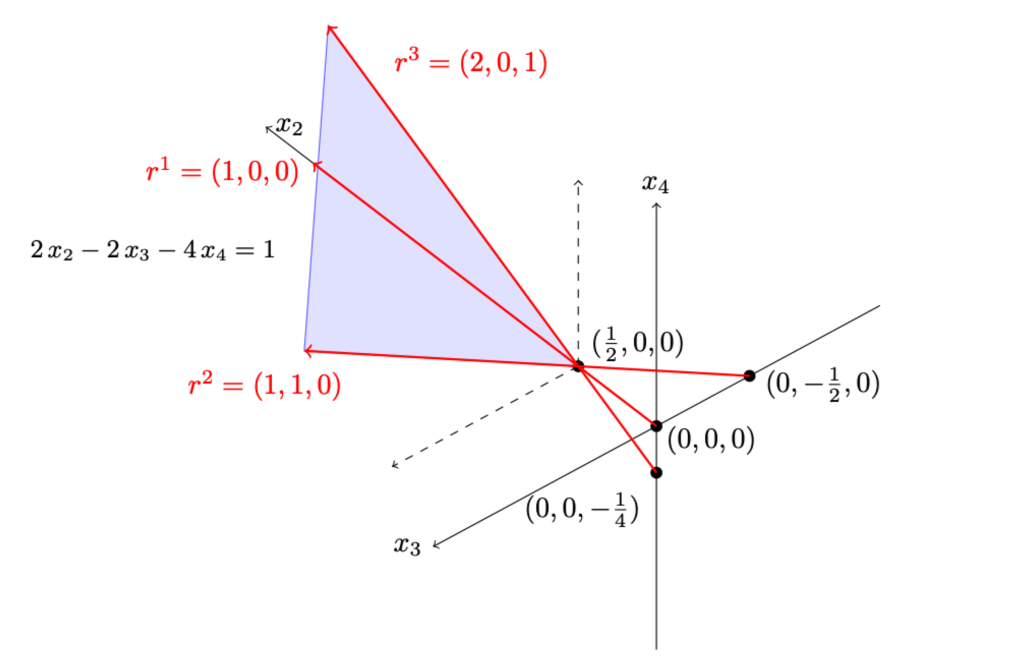


Figure 2: The half-space $2x_2 - 2x_3 - 4x_4 \geq 1$ in the (x_2, x_3, x_4)-axes with extreme rays.

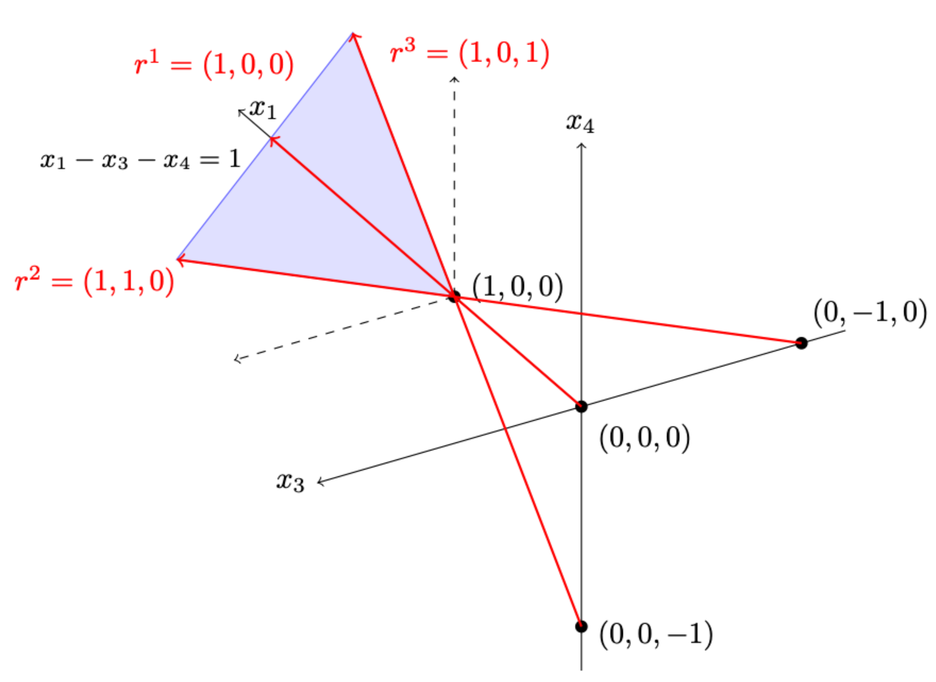


Figure 1: The half-space $x_1 - x_3 - x_4 \geq 1$ in the (x_1, x_3, x_4)-axes with extreme rays.

The NEEC cut and depth of cut

Negative Edge Extension Convexity (NEEC) cut:

$$\alpha_j = \max\{d^t_j\}, \forall j \in \mathcal{J}^+, \qquad \alpha_k = \max\left\{-\max\{d^t_j\} \min\left\{\frac{|d^t_k|}{d^t_j}\right\}\right\}, \forall k \in \mathcal{J}^-$$

Biextremal optimization problem:

Maximize distance from origin to the nearest point on the cut

$$\max_{\alpha \in S^\#} \min\{\|x\| : \alpha x \geq 1, x \in \mathbb{R}^n_{\geq 0}\}$$

For any monotonic norm $\|\cdot\|$ the NEEC cut is the global optimum.

$$\nu(\alpha) := \min\{\|x\| : \alpha x \geq 1, x \in \mathbb{R}^n_{\geq 0}\}$$

$$\nu(\alpha^{NEEC}) \geq \nu(\alpha), \forall \alpha \in S^\#_0$$

The log transformed reverse polar ($D^\#$)

Homogenize disjunctive constraints

$$S^t_0 = \{x \in \mathbb{R}^n_{\geq 0} : d^t x - d^t_0 \geq 0\}$$

$$S^\#_0 = \left\{ \begin{array}{c} \alpha \in \mathbb{R}^{n+1} : \min\left\{\frac{|d^t_k|}{d^t_j}\right\} \alpha_j + \alpha_k \geq 0, \\ \forall (j,k) \in \mathcal{J}^+ \times (\mathcal{J}^- \cup \{0\}) \end{array} \right\}$$

Log transform of the reverse polar

$$S^\#_0 \cong D^\# = \left\{ \begin{array}{c} \delta \in \mathbb{R}^{n+1} : \delta_k - \delta_j \leq c_{jk}, \\ \forall (j,k) \in \mathcal{J}^+ \times (\mathcal{J}^- \cup \{0\}) \end{array} \right\}$$

$$c_{jk} = \ln(\min\{|d^t_k|/d^t_j\}), \qquad \delta_i = \ln(|\alpha_i|), \forall i[n]$$

Enumerating the Vertices of ($D^\#$)

Goal: Generate facet-defining inequalities for cl conv(S) by enumerating D# vertices.

Problem: $D^\#$ can be highly degenerate giving us redundant cuts.

Theorem: There exists a two-term simple disjunction such that $D^\#$ has a linear number of vertices and an exponential number of basic feasible solutions.

Strategy: Don't enumerate basic feasible solutions. At a vertex enumerate all adjacent polyhedral edges.

The Auxiliary Graph G_η

Definition

- Auxiliary graph: $G_\eta(V, E)$
- $V = \mathcal{J}^+ \cup (\mathcal{J}^- \cup \{0\})$,
- $E = \{k \to j : \delta_k - \delta_j = c_{jk}\}$
- Slack: $\lambda_{jk} = \delta_j - \delta_k + c_{jk}$

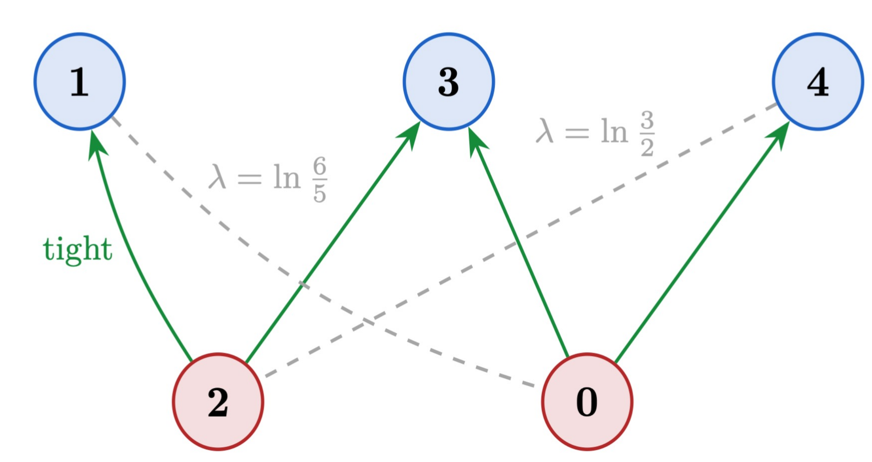
Example

$(5x_1 - 3x_2 + x_4 - x_0 \geq 0)$
$\vee$
$(3x_1 - x_2 + 2x_3 - 3x_4 - x_0 \geq 0)$
$\vee$
$(4x_1 - 6x_2 + 4x_3 - 2x_4 - x_0 \geq 0)$
$\vee$
$(2x_1 - 2x_2 - 2x_3 - x_0 \geq 0)$

Partition: $\mathcal{J}^+ = \{1,3,4\}$, $\mathcal{J}^- = \{0,2\}$

Vertex:

$\delta = (\ln(5), \ln(5/3), \ln(4), \ln(1), \ln(1))$

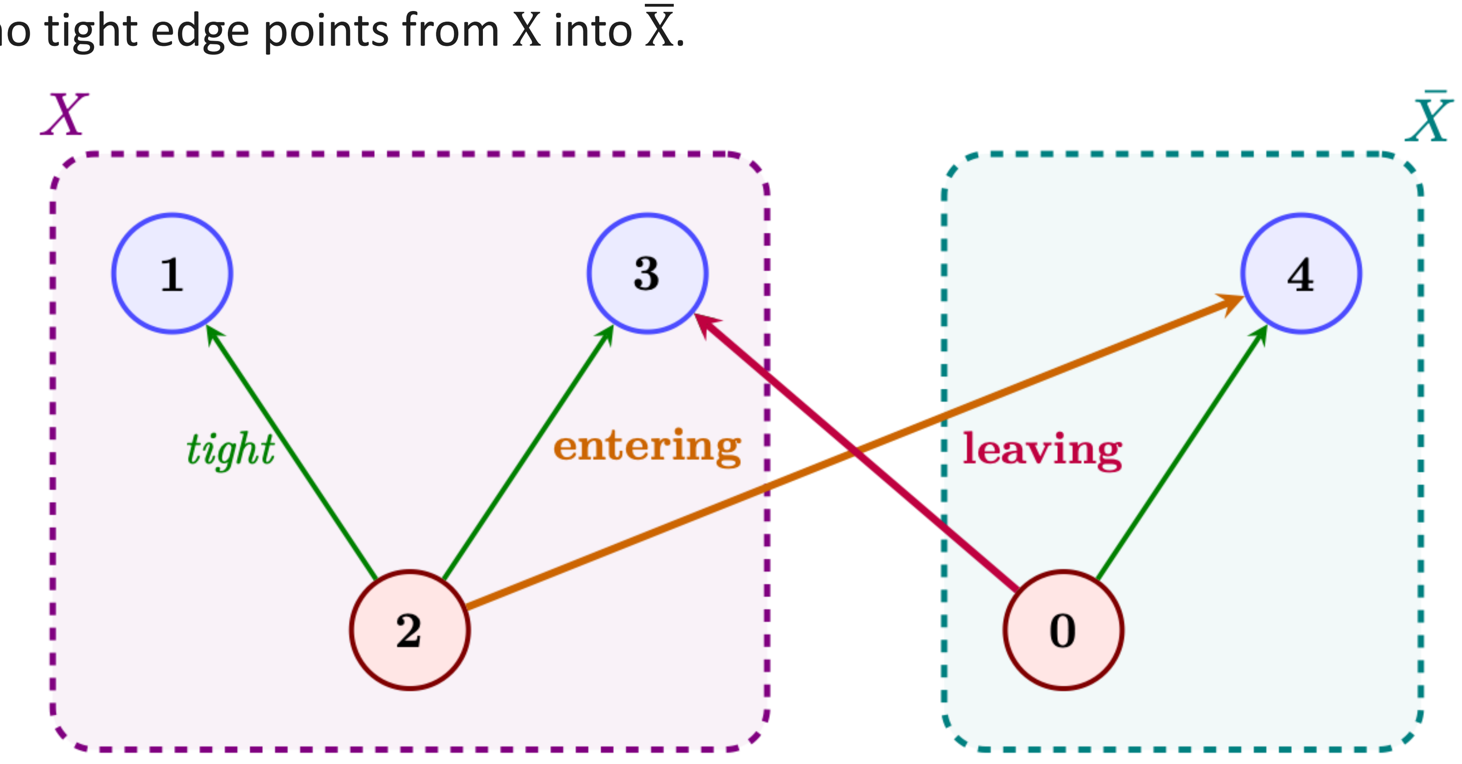


Adjacent polyhedral edges by graph cut $(X, \bar{X})$

(1) both X and $\bar{X}$ induce connected undirected subgraphs

(2) the edges of between them form a minimal disconnecting set

(3) no tight edge points from X into $\bar{X}$.



Enumeration procedure

1. Find a starting vertex η
2. Build G_η
3. Enumerate all graph cuts of G_η satisfying (1)-(3)

→ Adjacent vertex

4. Calculate new vertex $\eta *$ from graph cut
5. Ignore $\eta *$ if already found
6. Update vertex $\eta \leftarrow \eta *$

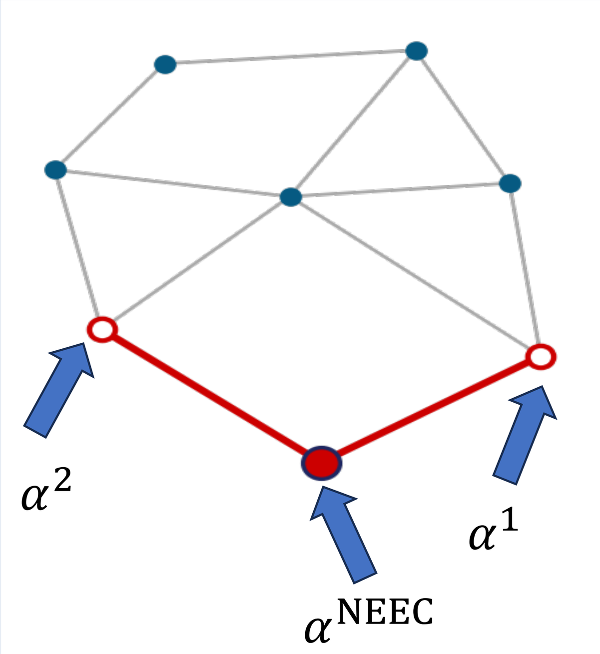
→ Filter

7. Repeat steps **2 – 5** until no new vertex found

→ Terminate

Complexity: This procedure is polynomial in the input size and number of vertices

Cut production in order of depth



1-skeleton of D#

Selection rule

- Start at the NEEC (global optimum)
- Among adjacent vertices select largest $\nu(\alpha)$
- $\alpha^{n+1} = \text{argmax}\{\nu(\alpha) : \alpha \text{ adjacent to some } \alpha^j, j \leq n\}$
- Cuts generated in non-increasing sequence
  - $\nu(\alpha^{NEEC}) \geq \nu(\alpha^1) \geq \nu(\alpha^2) \geq \cdots \geq \nu(\alpha^k)$