K-Adaptability in Stochastic Optimization with Recourse

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Motivation

Many real-world decisions must be made under **uncertain parameters**. We seek **near-optimal**, **feasible**, and **interpretable** solutions that can be enacted quickly once uncertainty is revealed. Several approaches exist:

- **Wait-and-see:** re-solve from scratch after observing uncertainty, optimal, but too slow and solutions vary wildly across scenarios.
- **Static stochastic:** commit to a single solution offline, interpretable, but can be highly suboptimal or even infeasible.
- **K-adaptability (K-ST):** precompute K full solutions, pick best online, but requires feasibility for *all* scenarios.

Our approach (K-SR) combines the best of both: precompute K binary plans offline, then adapt continuous recourse variables online.

K-Adaptability with Recourse

Offline

Let ω be a random parameter and $X = \{x \in \{0,1\}^p : Ax \geq b\}$. Solve

$$\min_{\mathcal{F} \subseteq X,\, |\mathcal{F}| \leq K} \mathbb{E}_\omega \left[\min_{x \in \mathcal{F}} \{ c(\omega)^\top x + f(x,\omega) \} \right]$$

where $f(x,\omega) = \min\{ d(\omega)^\top y : T(\omega)x + W(\omega)y \geq h(\omega) \}$

Online (after observe ω)

For each $x^k \in \mathcal{F}$, $k = 1, \ldots, K$, solve LP:

$$y^k \in \arg\min\{ d(\omega)^\top y : (x^k, y) \in F(\omega) \}$$

Implement (x^k, y^k) minimizing $c(\omega)^\top x^k + d(\omega)^\top y^k$.

Compact Formulation

Decision Variables:

- Recourse variables: y^{is}, $\forall s \in [N]$, $i \in [K]$
- K representative binary vectors: x^i, $i \in [K]$
- Assignment vars: $w_i^s \in \{0,1\}$, $i \in [K]$, $s \in [N]$

First Formulation *(Malaguti et al. 2022)*

$$\min_{x,w,y} \frac{1}{N} \sum_{i \in [K]} \sum_{s \in [N]} \left((c^s)^\top x^i + (d^s)^\top y^{is} \right) w_i^s$$

$$\begin{aligned}
\text{s.t.} \quad & T^s x^i + W^s y^{is} \geq h^s, && \forall s \in [N],\, i \in [K], \\
& \mathbf{1}^\top w^s = 1,\ w^s \in \{0,1\}^K, && \forall s \in [N], \\
& x^i \in X, && \forall i \in [K].
\end{aligned}$$

New Compact Formulation

Additional decision variables: q^s, $s \in [N]$

$$\min_{x,w,y,q} \frac{1}{N} \sum_{i \in [K]} \sum_{s \in [N]} \left((c^s)^\top q^s + (d^s)^\top y^{is} \right) w_i^s$$

$$\begin{aligned}
& T^s q^s + W^s y^{is} \geq h^s, && \forall s \in [N],\, i \in [K] \\
& \mathbf{1}^\top w^s = 1,\ w^s \in \{0,1\}^K, && \forall s \in [N] \\
& x^i \in X, && \forall i \in [K], \\
& (w_i^s - 1)\mathbf{1} \leq x^i - q^s \leq (1 - w_i^s)\mathbf{1}, && \forall i \in [K],\, s \in [N]
\end{aligned}$$

Assignment Formulation (AF)

Decision variables:

- $\sigma_v \in \{0,1\}$: takes 1 if $v \in X$ is a representative
- $\rho_{sv} \in \{0,1\}$: takes 1 if v represents scenario s

$$\min_{\sigma,\rho} \frac{1}{N} \sum_{s \in [N]} \sum_{v \in X} f(\omega^s, v)\, \rho_{sv}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{v \in X} \rho_{sv} = 1, && \forall s \in [N], \\
& \rho_{sv} \leq \sigma_v, && \forall s \in [N],\, v \in X, \\
& \sum_{v \in X} \sigma_v \leq K, && \\
& \sigma_v \in \{0,1\}, && \forall v \in X, \\
& \rho_{sv} \in \{0,1\}, && \forall v \in X,\, s \in [N].
\end{aligned}$$

Heuristic/Exact Methods

Scenario-Based Heuristic

- Let $\tilde{\omega}^s$ for $s = 1, \ldots, M$ be a sample of random variables.
- For each s, find $x^s \in \arg\min\{ f(\tilde{\omega}^s, x) : x \in X \}$.
- Let $\mathcal{L} = \{ x^s : s = 1, \ldots, M \}$ and replace X with tractable $\mathcal{L}$ in the AF.

$$\min_{\sigma,\rho} \frac{1}{N} \sum_{s \in [N]} \sum_{v \in \mathcal{L}} f(\omega^s, v)\, \rho_{sv}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{v \in \mathcal{L}} \rho_{sv} = 1, && \forall s \in [N], \\
& \rho_{sv} \leq \sigma_v, && \forall s \in [N],\, v \in \mathcal{L}, \\
& \sum_{v \in \mathcal{L}} \sigma_v \leq K, && \\
& \sigma_v \in \{0,1\}, && \forall v \in \mathcal{L}, \\
& \rho_{sv} \in \{0,1\}, && \forall v \in \mathcal{L},\, s \in [N].
\end{aligned}$$

- In general, restricting to $\mathcal{L}$ may not be sufficient for optimality even as $M \to \infty$.
- Restricting to $\mathcal{L}$ is sufficient as $M \to \infty$ if only the objective is random and support of the random vector is convex.

Column & Constraint Generation

- **Master problem:** includes only variables and constraints in $X' \subseteq X$.
- **Pricing problem:** look for $v \in X \setminus X'$ decreasing the objective; add corresponding variables and constraints to the master.

Solution v improves if total savings over current coverage costs exceed μ^*:

$$\sum_{s \in [N]} \max\{0,\, \lambda_s^* - f(\omega^s, v)\} > \mu^*$$

Pricing Formulation:

$$\max_{x,y,\pi,q} \sum_{s \in [N]} \left(\lambda_s^* \pi_s - (c^s)^\top q^s - (d^s)^\top y^s \right)$$

$$\begin{aligned}
& x \in X,\ \pi \in \{0,1\}^N, && \\
& T^s q^s + W^s y^s \geq h^s \pi_s, && \forall s \in [N], \\
& A q^s \geq b \pi_s, && \forall s \in [N], \\
& 0 \leq q^s \leq \mathbf{1}\pi_s, && \forall s \in [N], \\
& -\mathbf{1}(1-\pi_s) \leq x - q^s \leq \mathbf{1}(1-\pi_s), && \forall s \in [N].
\end{aligned}$$

Experiment Setup

- **Instances:** cap91 (25 facilities, 50 customers) and cap111 (50 facilities, 50 customers).
- Customer demands are parametric.
- Three distributions for customer demands: Gaussian with low variance, Gaussian with high variance, and Bernoulli×Gaussian.
- Computational effort measured with Gurobi Work Units (GWU), a solver-independent measure of effort.

Solution Quality

Average gap (%) relative to wait-and-see (S=25, L=300):

	cap91			cap111		
K	Low	High	Bern.	Low	High	Bern.
1	4.03	7.38	23.63	2.32	4.10	22.01
2	1.97	3.49	12.72	1.27	2.09	12.38
3	1.12	2.30	10.49	0.76	1.43	9.28
4	0.69	1.57	8.88	0.46	0.90	7.45
5	0.48	1.19	7.55	0.36	0.66	6.62
10	0.10	0.31	3.67	0.08	0.17	3.71

Heuristic gap (%) and AF work units (GWU) (S=25, L=300):

	cap91		cap111	
K	Gap	GWU	Gap	GWU
2	0.41	17514	0.78	5092
3	0.70	4849	1.11	9313
4	0.63	10561	0.87	6004
5	0.56	7582	0.91	11118
10	0.18	2260	0.89	5866

Compact Formulation Results

Optimality gap (%) at 30000 GWU: *(C1 = Malaguti et al. 2022)*

		cap91		cap111	
S	K	C1	New	C1	New
25	2	25.95	0.00	35.72	3.47
	3	33.08	4.05	52.92	6.67
	4	32.12	4.66	56.21	6.19
50	2	38.18	4.70	79.66	6.34
	3	48.18	7.11	75.57	7.56
	4	48.56	8.94	80.83	6.66

New compact formulation achieves significantly lower MIP gaps than C1 across all settings.

Conclusion and Future Work

- The heuristic is fast and produces solutions of very high quality.
- Our new compact formulation performs better than extensions of prior formulations for the static case.
- Our column and constraint generation approach shows promise but the pricing problem is hard to solve.
- Future work focuses on efficiently solving the pricing problem for larger instances.