Boxing Points in One Dimension

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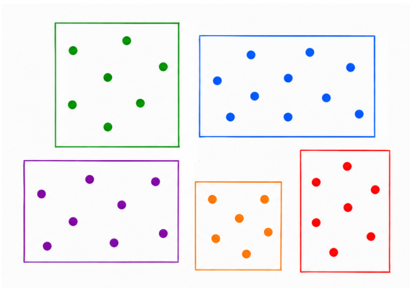
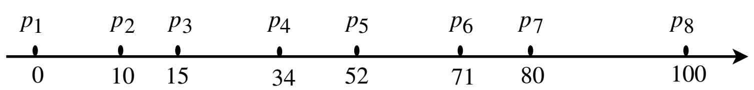
Clustering via hyper-rectangles

Goal: put points in boxes to maximize the weight while minimizing size of the hyper-rectangles.



Example of boxing in 1D: $p = [0,10,15,34,52,71,80,100]$



1. $\mathrm{wd}(\{0,1,2,3,4,5,6,7,8\}) = p_8 - p_1 = 100$.
2. $\mathrm{wd}(\{5\}) = p_5 - p_5 = 0$.
3. $\mathrm{wd}(\{2,4,5\}) = p_5 - p_2 = 42$.
4. $\mathrm{wd}(\{\emptyset\}) = 0$

- The **width** of a set $S \subseteq [n]$ is $\mathrm{wd}(S) := \max\{0,\ u(S) - \ell(S)\}$.

MIP formulation

$$\begin{aligned} \max\quad & c^\top z - w \\ \text{subject to}\quad & B \geq w \geq \max\{0,\ u(z) - \ell(z)\} \\ & z \in \{0,1\}^n \end{aligned}$$

We study the convex hull:

$$\mathrm{conv}\left\{(w,z) \in \mathbb{R} \times \mathbb{Z}^n : B \geq w \geq \max\{0,\ u(z) - \ell(z)\}, z \in \{0,1\}^n\right\}$$

Related work

- Consider the following sets:

$$U = \left\{(u,z) \in \mathbb{R} \times \mathbb{Z}^n : u \geq p_i z_i\ \forall i \in [n], z \in \{0,1\}^n\right\}$$
$$L = \left\{(\ell,z) \in \mathbb{R} \times \mathbb{Z}^n : (1-\ell) \geq (1-p_i) z_i\ \forall i \in [n], z \in \{0,1\}^n\right\}$$

U and L are mixing sets, and $\mathrm{conv}(U \cap L) = \mathrm{conv}(U) \cap \mathrm{conv}(L)$.

- If $\overline{W} = \left\{(w,z) \in \mathbb{R} \times \{0,1\}^n : w \geq u(z) - \ell(z), z \in \{0,1\}^n\right\}$, $\mathrm{conv}(\overline{W})$ is given by the Lovász extension by submodularity of $u(z) - \ell(z)$.

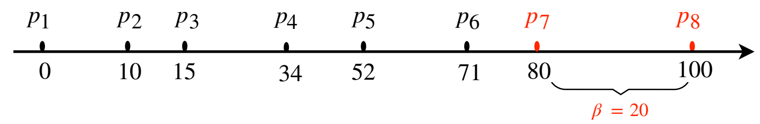
Main results

Let $W^\infty = \mathrm{conv}\left\{(w,z) \in \mathbb{R} \times \mathbb{Z}^n : w \geq \max\{0,\ u(z) - \ell(z)\}, z \in \{0,1\}^n\right\}$
and $W^B = \mathrm{conv}\left\{(w,z) \in \mathbb{R} \times \mathbb{Z}^n : B \geq w \geq \max\{0,\ u(z) - \ell(z)\}, z \in \{0,1\}^n\right\}$
We characterize W^∞ and W^B

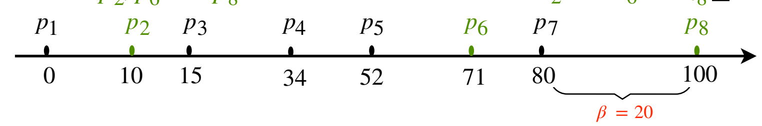
Generating facets for W^∞

- Trivial facets: $w \geq 0,\ 1 \geq z_i \geq 0\ \forall i \in [n]$.
- Non-trivial facets: $w - \alpha^\top z \geq -\beta$, where $\alpha \geq 0,\ \beta > 0$.

1. Pick two anchors, p_7 and p_8, set $\beta = p_8 - p_7 = 20$.

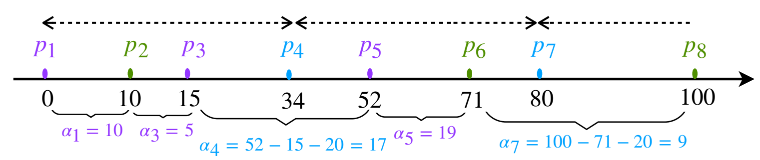


2. Pick p_2, p_6 and p_8 as leaves. $\Longrightarrow$ seed $w - 20z_2 - 20z_6 - 20z_8 \geq -20$.



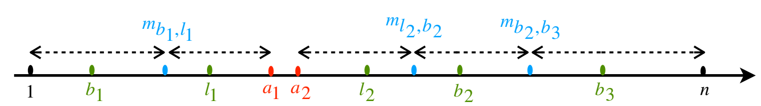
3. Set p_4 and p_7 as middle points. Lifting on both sides of each leaf.

$$w - 10z_1 - 20z_2 - 5z_3 - 17z_4 - 19z_5 - 20z_6 - 9z_7 - 20z_8 \geq -20$$



- In general, let B be leaf set:

$$w - \sum_{i \in B} \beta z_i - \sum_{i \notin B} \alpha_i z_i \geq -\beta, \quad \alpha_i = \begin{cases} p_i - p_{i-1} & \text{if } i \text{ is lifted left to right} \\ p_{i+1} - p_i & \text{if } i \text{ is lifted right to left} \\ p_{i+1} - p_{i-1} - \beta & \text{if } i \text{ is middle point} \end{cases}$$



Theorem 1: These lifting rules generate all facets of W^∞.

Proof Sketch

Fix a non-trivial facet. Consider the family of tight sets on this facet: $\mathcal{T} := \left\{S \subseteq [n] : \mathrm{wd}(S) - \alpha(S) = -\beta\right\}$.

Crossing Lemma:
1. If $S, T \in \mathcal{T}$ and $S \cap T \neq \emptyset$, then both $S \cap T$ and $S \cup T$ are in $\mathcal{T}$.
2. Moreover, $\alpha(S) + \alpha(T) = \alpha(S \cup T) + \alpha(S \cap T)$.

Uncrossing: If $\mathcal{T}$ satisfies the Crossing Lemma, then we can extract a maximal Laminar family $\mathcal{L} \subseteq \mathcal{T}$ by replacing $S, T \in \mathcal{T}$ where $S \cap T \neq \emptyset$ by $S \cup T$ and $S \cap T$.

Lemma 2: Let $\hat{\mathcal{T}} := \left\{(\mathrm{wd}(S), z(S)) : S \in \mathcal{T}\right\}$ and $\hat{\mathcal{L}} := \left\{(\mathrm{wd}(S), z(S)) : S \in \mathcal{L}\right\}$. Then $\mathrm{aff}(\hat{\mathcal{T}}) = \mathrm{aff}(\hat{\mathcal{L}})$.

Corollary 3: $\hat{\mathcal{L}}$ contains $n+1$ affinely independent tight points on the facet that uniquely determines α and β.

Construct a directed forest $F = (\mathcal{L}, A)$ where $(S,T) \in A$ if S is the unique minimal set that contains T as a subset. Root each tree at the maximal sets.

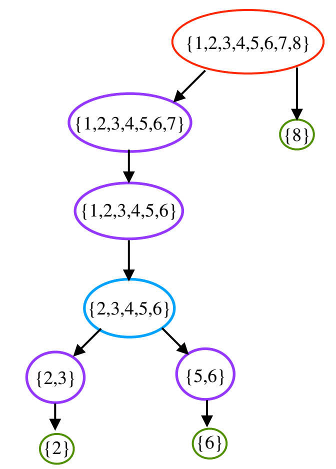


Fig. 1: The Laminar tree for the example facet

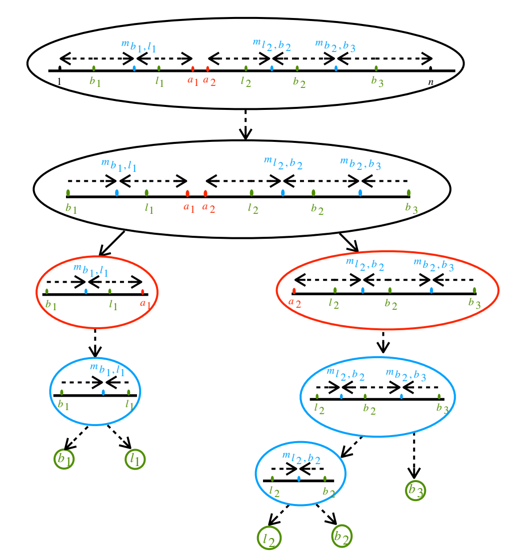


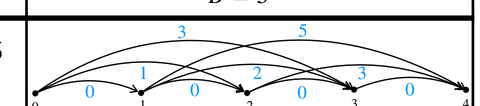
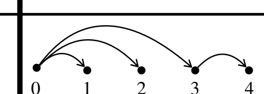
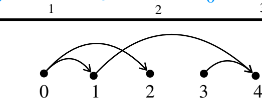
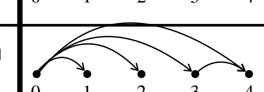
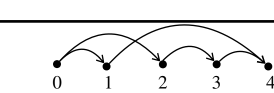
Fig. 2: The Laminar tree for a general facet

Generating facets for W^B

- Trivial facets: $B \geq w \geq 0,\ 1 \geq z_i \geq 0\ \forall i \in [n]$.
- Non-trivial facets:
  1. **Clique facets**: $z(Q) \leq 1$, for maximal incompatible points Q.
  2. $w - \alpha^\top z \geq -\beta$, where $\alpha \geq 0,\ \beta > 0$.

Construct a graph $G = (V, A)$: $V = \{0,1,\ldots,n\}$, $(i,j) \in A$ if $p_j - p_{i+1} \leq B$ and $w(i,j) = p_j - p_{i+1}$. Let T be a spanning tree of G.

Example: $p = [1,2,4,7]$

	$B = \infty$	$B = 5$
G	Complete graph on 5 vertices	
T		
Add an arc		
Facet	$w - z_1 - 2z_2 - 3z_3 - 3z_4 \geq -3$	$w - 4z_1 - z_2 - 4z_3 - 4z_4 \geq -4$

References

1. Baumann, F., Berckey, S., and Buchheim, C. (2013). Exact Algorithms for Combinatorial Optimization Problems with Submodular Objective Functions, pages 271-294. Springer Berlin Heidelberg, Berlin, Heidelberg.
2. Günlük, O. and Pochet, Y. (2001). Mixing mixed-integer inequalities. Mathematical Programming, 90(3):429-457.
3. Lovász, L. (1983). Submodular functions and convexity, pages 235-257. Springer Berlin Heidelberg, Berlin, Heidelberg.