Parametric Models for Minimal Functions and Optimizing Cuts for Mixed Integer Programs

Acadia Larsen

University of California, Davis

The Infinite Group Problem and the Space of Minimal Functions

Definition 1 (Gomory, Johnson, 1972)**.** Let $f \in \mathbb{R}^q$ and consider the problem

$$-f + \sum_{r \in \mathbb{R}^q} r x_r \in \mathbb{Z}^q \tag{1}$$

where x has finite support and $x_r \in \mathbb{Z}_+$. Let M_f denotes the set of all feasible solutions to the problem (1). This known as the (multi-row) Infinite Group Problem. A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is said to be *valid* for the problem (1) if $\pi \geqslant 0$ and $\sum_{r \in \mathbb{R}^q} \pi(r) x_r \geqslant 1$ for all $x \in M_f$. A valid function π is said to be *minimal* if there is no distinct valid function π^1 such that $\pi^1 \leqslant \pi$.

Definition 2. We say that a *proper breakpoint* of a piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is a point λ in the domain of f such that f is C^0 at λ but not C^1 at λ. Let PBKPT_f be the set of proper breakpoints of f. A *virtual breakpoint* is a distinguished point λ in the domain of f such that f is C^1 at λ. A *breakpoint* is a virtual or proper breakpoint. A *breakpoint sequence* of a f is a non-decreasing sequence BKPT_f of breakpoints such that PBKPT_f is contained in BKPT_f The *multiplicity* of a breakpoint in a breakpoint sequence is the number of occurrences of the breakpoint in the breakpoint sequences, denoted $\text{MULT}(\lambda_i)$.

Definition 3. For integers $n \geqslant 2$, define $\text{PWL}_{\leqslant n}$ as the space of $\mathbb{Z}$ periodic continuous piecewise linear functions with $|\text{PBKPT}|_{f|_{[0,1)}}| \leqslant n$ and $f(0) = f(1) = 0$ and let $\Pi^{\text{min}}_{\leqslant n} = \{\pi \in \text{PWL}_{\leqslant n} : \exists$ one row $(q = 1)$ igp such that π is minimal for the igp$\}$.

Definition 4. Suppose that $\{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{n-1}\} = b$ is a breakpoint sequence of $\pi \in \text{PWL}_{\leqslant n}$ in the interval $[0, 1)$. Let $B = \text{BKPT}_\pi$ be the set of all breakpoints of π. We define a two dimensional polyhedral complex, $\Delta\mathcal{P}_B$, with faces $F = F(I, J, K) = \{(x, y) \in \mathbb{R}^2 | x \in I, y \in J, x + y \in K\}$ where $I, J, K \in \{[\lambda_i + t, u\lambda_i + (1-u)\lambda_{i+1} + t] : 0 \leqslant i \leqslant n-1, t \in \mathbb{Z}, u \in \{0, 1\}\}$. $\mathcal{P}_b$ is the restriction of $\mathcal{P}_B$ to $[0, 1]^2$.

Theorem 5 (Basu, Hildebrand, Köppe, 2015, rephrased)**.** *A piecewise linear function $\pi \in PWL_{\leqslant n}$ is minimal if and only if the following conditions hold:*

(i) Subadditivity test: For all $F \in \Delta\mathcal{P}_B$ with $F \subseteq [0, 1]^2$, and $(x, y) \in$ vert(F), we have have that $\Delta\pi(x, y) \geqslant 0$.

(ii) Symmetry test: $\pi(f) = 1$ and for all $F \in \Delta\mathcal{P}_B$ with $F \subseteq \{(x, y) \in [0, 1]^2 | x + y \equiv f \bmod 1\}$ we have that $\Delta\pi(x, y) = 0$ for all $(x, y) \in$ vert(F).

Optimal Cut Generation from Parameterized Cut Generating Functions

Suppose an mixed integer set

$$\{Ax = b; x_j \in \mathbb{Z}, j \in I \subset [n], x \geqslant 0\} \tag{2}$$

is given with B be a indices of feasible basis of the LP relaxation (2) and N be a set of indices corresponding to the non basic variables. The corner polyhedron is the convex hull of relaxation given by $x_i = \overline{b}_i - \sum_{j \in N} \overline{a_{ij}} x_j : i \in B; x_i \in \mathbb{Z}, i \in I; x_j = 0, j \in N$. Let ψ_π be the superlinear portion of $\pi \in \Pi^{\text{min}}_{\leqslant k}$ about 0 extended to $\mathbb{R}$. Let a row $i \in B$ be fixed such that $\overline{b}_i \notin \mathbb{Z}$. If $\pi(\overline{b_i}) = \pi(f) = 1$, we have

$$\sum_{j \in N \cap I} \pi(\overline{a_{ij}}) x_j + \sum_{j \in N \cap C} \psi_\pi(\overline{a_{ij}}) \geqslant 1 \tag{3}$$

is a intersection cut for the corner polyhedron. A *cut space generated from a parameterized set S of minimal functions* denoted $\text{CutSpace}_{i,S}$ is the set of all intersection cuts of the form in (3) such that $\pi_p \in \phi(S \cap U), (\phi, U) \in \mathcal{A}_k$, (Theorem 9) $S \subset \mathbb{R}^{2k}$, and $\pi_p \in \Pi^{\text{min}}_{\leqslant k}$. Let $S^{\text{min}}_{\leqslant k} = \bigcup_{\alpha \in \mathbb{A}_k} S_\alpha$ (Corollary 11). Suppose that $s : \text{CutSpace}_{i,S} \to \mathbb{R}$ is a cut scoring function that is smooth enough on the cut space. The optimization problem

$$\max s(\gamma) s.t. \gamma \in \text{CutSpace}_{i, S^{\text{min}}_{\leqslant k}} \tag{4}$$

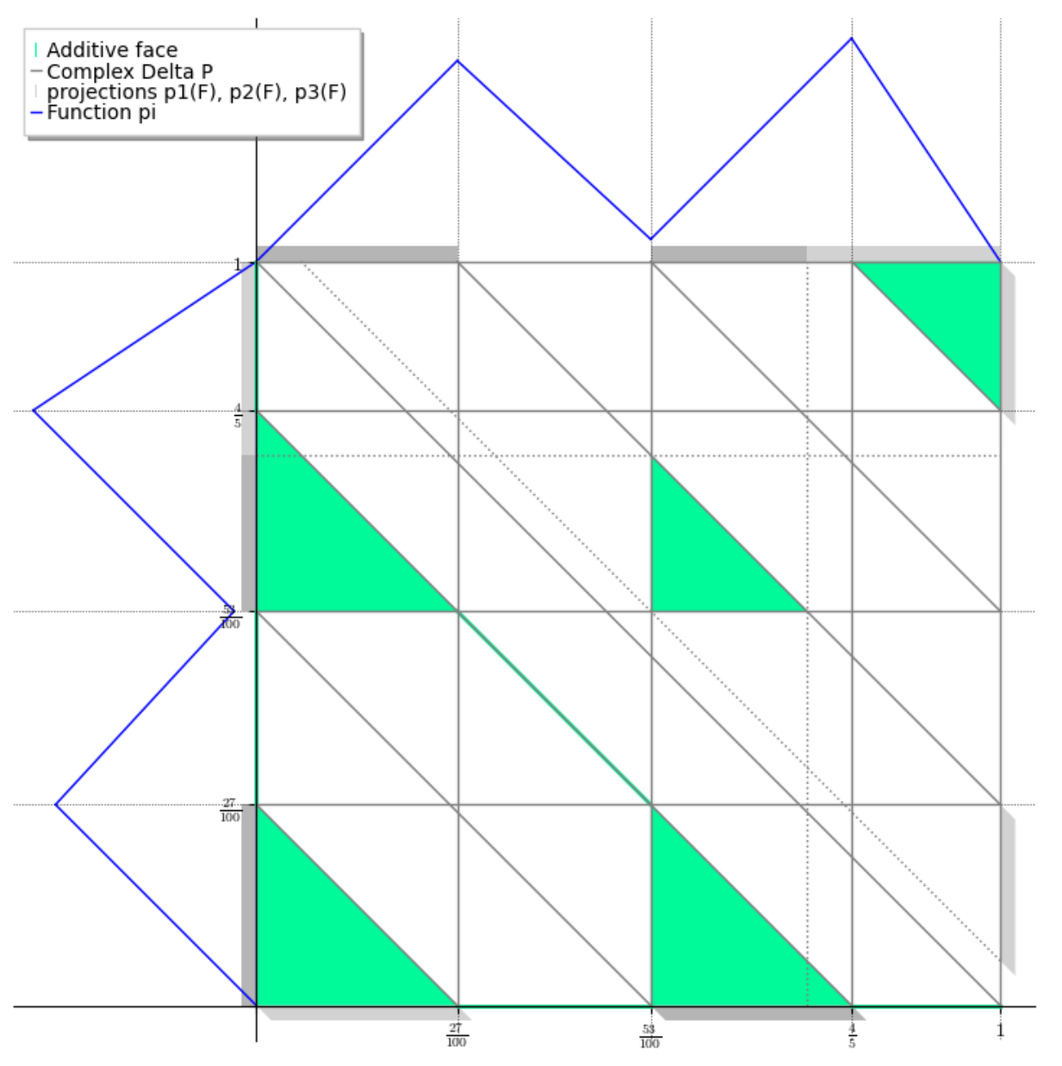
is called *the full space cut optimization problem*. When symmetrized row data is used (Proposition 12), the specialization

$$\max s(\gamma) s.t. \gamma \in \text{CutSpace}_{i, V_{\text{sym}(\overline{a_i}), \text{f index}}} \tag{5}$$

is called *the breakpoint as parameters cut optimization problem.* When the solution that is considered is a cut generation function, we refer to this as a *cut generation problem.*

`parametricCutGeneration` implements an alpha version of a solver for cut generation problems and a python interface to `scip` via `pyscipopt` for cut optimization problems.

Example: Let s be the steepest direction cut score with data $\overline{a_i} = [3.27, 4.66, 5.53, .56]$, $\overline{c_N} = [-1.2, -4.4, -5.6, -.1]$, $\overline{b_i} = 1.8$



The solution to the cut generating problem in (5) using the data given.

Planned HPC Experiments: Compare using gmic $(n = 2)$ to breakpoint as parameters cut generation problem with at most $2j$ breakpoints $2 \leqslant j \leqslant compute\ budget$ by adding cuts at the root node.

Parametric Models of Continuous Piecewise Linear Functions

Definition 6. Let X be a paracompact Hausdorff topological space. An *n-dimensional manifold with corners* is X equipped with a maximal n-dimensional atlas $\mathcal{A}$ with corners (model space is $[0, \infty)^k \times \mathbb{R}^{n-k}$).

Definition 7. Let $\lambda_l, \lambda_u, \lambda_c, \in [0, 1]$ and suppose that $\lambda_l < \lambda_u$ with $\lambda_c \in (\lambda_l, \lambda_i)$. A $\mathbb{Z}$-periodic hat function on the interval (λ_l, λ_u) with center λ_c is a $\mathbb{Z}$-periodic function $h_{\lambda_l, \lambda_c, \lambda_u} : \mathbb{R} \to \mathbb{R}$ such that

$$h_{\lambda_l,\lambda_c,\lambda_u}(x) = \begin{cases} \frac{1}{\lambda_c - \lambda_l}(x \bmod 1 - \lambda_l) \text{ if } x \bmod 1 \in [\lambda_l, \lambda_c] \\ -\frac{1}{\lambda_u - \lambda_c}(x \bmod 1 - \lambda_u) \text{ if } x \bmod 1 \in [\lambda_c, \lambda_u] \\ 0 \text{ otherwise.} \end{cases}$$

Definition 8. Let $\mathcal{B}_n$ be the space of breakpoint sequences in $[0, 1)$ of length n and $\text{MULT}(0) = 1$. The parameterization map $\zeta_n : \mathcal{B}_n \times \text{PWL}_{\leqslant n} \to \mathbb{R}^{2n}$ sends

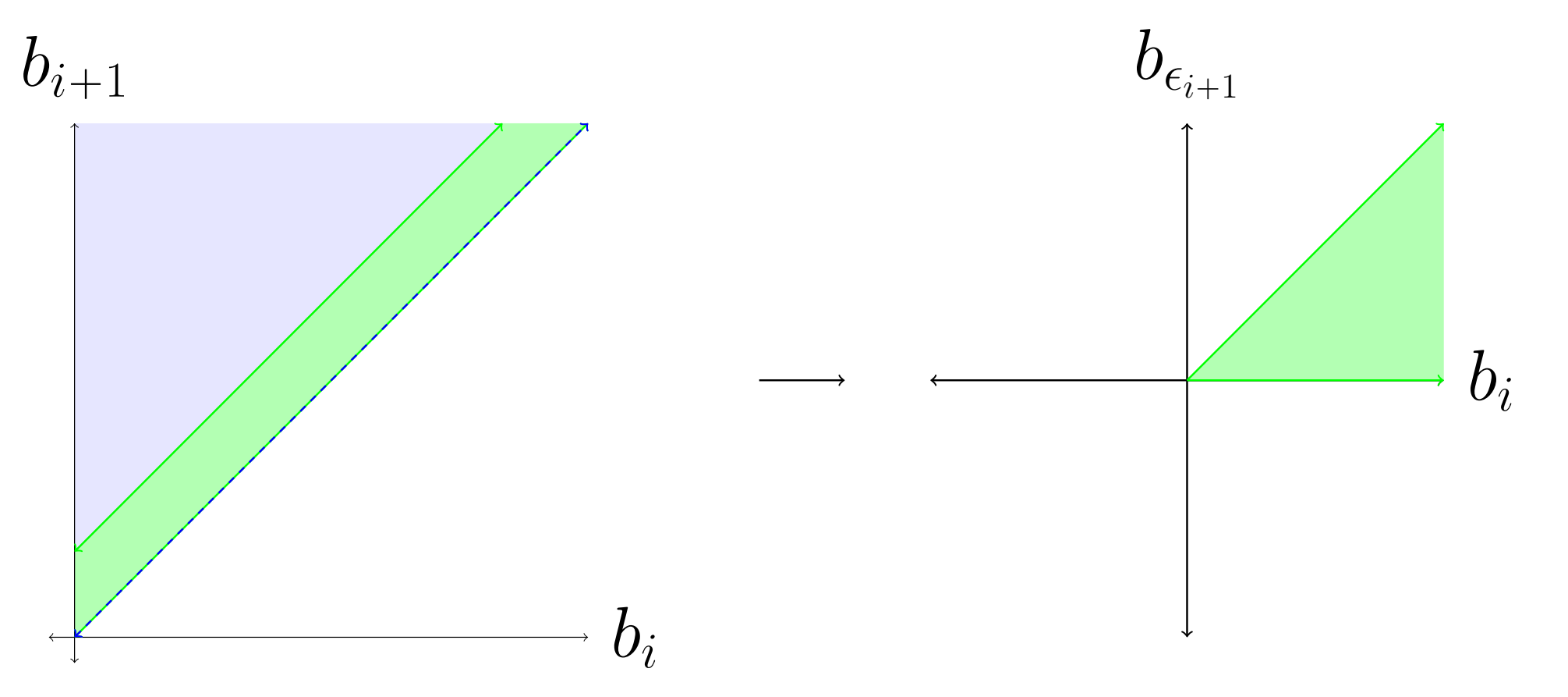
$$\begin{aligned}(\text{bkpt}, f) &\to (\lambda_0, \ldots, \lambda_{n-1}, \frac{f(\lambda_0)}{\text{MULT}(\lambda_0)}, \ldots, \frac{f(\lambda_{n-1})}{\text{MULT}(\lambda_{n-1})}) \\ &= (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \\ &= (\lambda, \gamma) = (b, v) = (b_0, .., b_{n-1}, v_0, \ldots, v_{n-1}). \end{aligned} \tag{6}$$

Give $\zeta_n(\text{bkpt}, f) = (b, v)$, we use $f_{(b,v)}$ to a presentation of $f = \sum_{\lambda_c \in \text{bkpt}} \gamma_c h_{\lambda_l, \lambda_c, \lambda_u}(x)$ by $\mathbb{Z}$ periodic hat functions.

Theorem 9. *$\mathcal{B}_n \times PWL_{\leqslant n}$ is a manifold with corners with atlas $\mathcal{A}_n$.*

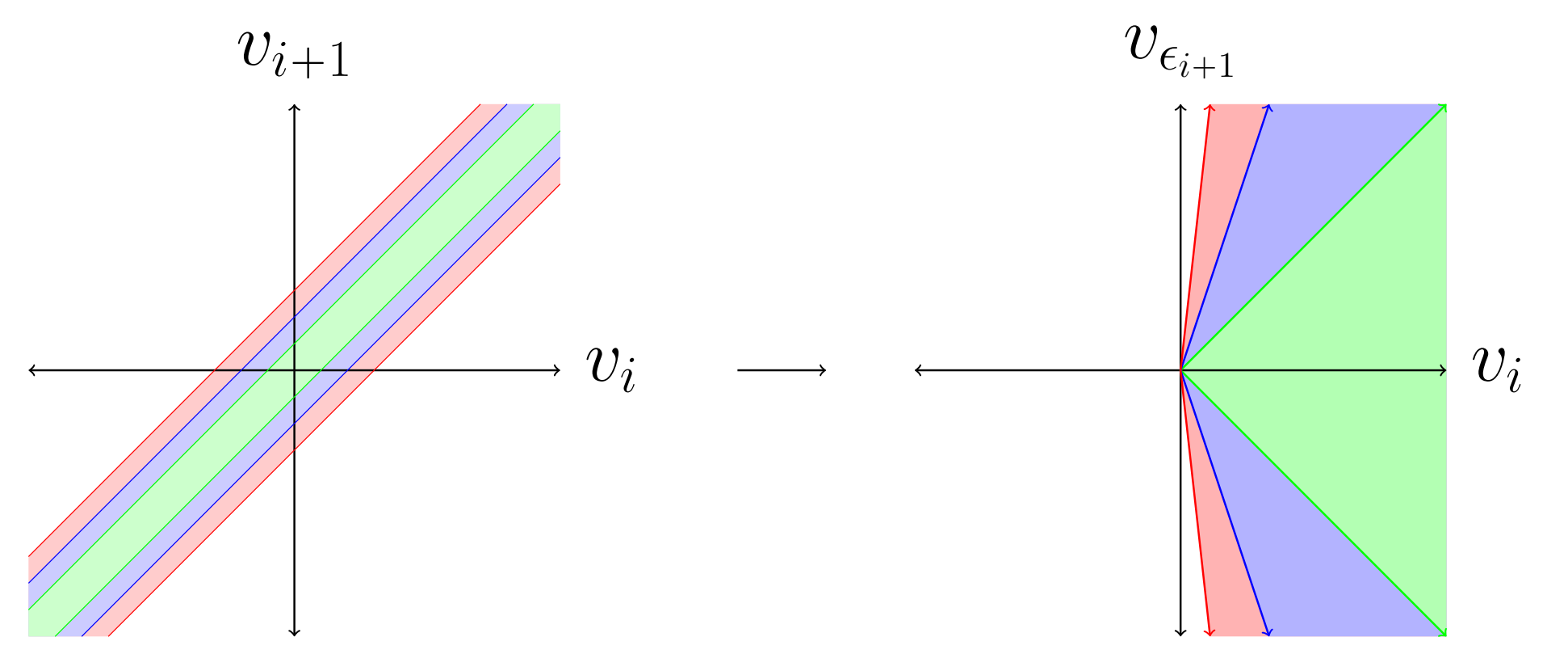
Note: NNC polyhedra can be used in place of the corner space $[0, \infty)^k \times \mathbb{R}^{n-k}$ for the definition manifolds with corners.

Examples of charts: $\phi_{\text{mult}=1}$ has a domain $b_i < b_{i+1}$ (light blue region) and sends parameters $b_i \to \lambda_i$, $v_i \to \gamma_i$ with $f = \sum_{i=1}^{k} \gamma_i h_{\lambda_{\lambda_{i-1}, \lambda_i, \lambda_{i+1}}}(x)$.



(left, light blue), domain of $\phi_{\text{mult}=1}$, (left, green strip) models when $b_{i+1} = \lambda_{i+1} = b_i + b_{\epsilon_{i+1}} \to b_i$ (right, green region) coordinates for $\phi_{M, b_{\epsilon_i}}$.

A chart $\phi_{M, b_{\epsilon_{i+1}}}$ $(M > 0$, fixed) is defined using the RHS pictures as coordinates with the constraints $0 \leqslant b_i \leqslant b_{\epsilon_{i+1}}$, $|v_{\epsilon_{i+1}}| \leqslant M b_{\epsilon_{i+1}}$ with $\lambda_j = b_j (j \neq i+1)$, $\lambda_{i+1} = b_i + b_{\epsilon_{i+1}}, \gamma_{i+1} = v_i + v_{\epsilon_{i+1}}$, and $f = \sum_{i=1}^{k} \gamma_i h_{\lambda_{\lambda_l, \lambda_i, \lambda_u}}(x)$.



(left) $|v_i - v_{i+1}| \leqslant M b_{\epsilon_{i+1}}$ for $M b_{\epsilon_{i+1}} = .3, .6, .9$
(right) $|v_{\epsilon_{i+1}}| \leqslant M b_{\epsilon_{i+1}}$ with $M = 1, 3, 9$, coordinates for $\phi_{M, b_{\epsilon_i}}$.

Semialgebraic Models for Minimal Functions

Let $\mathcal{V}_n \subseteq [0, 1]^n$ be the space of values such that for $b \in \mathcal{B}_n \subseteq [0, 1]^n$, $\pi_{(b,v)}$ is valid for an infinite group problem.

Theorem 10 (Semialgebraic Minimality Test)**.** *Let $n \geqslant 0$ and $f \in (0, 1)$ be fixed. There is a finite set of degree at most three polynomials $\mathcal{H}_{n,f} = \{h_i : h_i \in \mathbb{R}[\lambda_0, \ldots, \lambda_{n-1}, \gamma_0, \ldots, \gamma_{n-1}]\}$ such that for any fixed point $\overline{p} \in \mathcal{B}_n \times \mathcal{V}_n$, there is a finite index set I such that:*

$$\overline{p} \in BSA = \bigcap_{i \in I} \{x \in \mathbb{R}^n : h_i(x) \bowtie_i 0 \text{ and } h_i \in H\} \tag{7}$$

*where $\bowtie_i \in \{<, \leqslant, =, \geqslant, >\}$ and for any $p \in BSA$, π_p is ** if and only if $\pi_{\overline{p}}$ is ** and at most n breakpoints where ** is valid but not minimal, minimal, or not constructiable for a given igp.*

Observation 1. Let $v = (x, y)$ be a vertex in $\Delta\mathcal{P}_b$, then $\Delta\pi_{(b,v)}(x, y)$ is a piecewise rational function in terms of the breakpoint and value parameters.

Observation 2. Let $b, \overline{b}$ be breakpoint sequences such that $\Delta\mathcal{P}_b \cong \Delta\mathcal{P}_{\overline{b}}$. Then

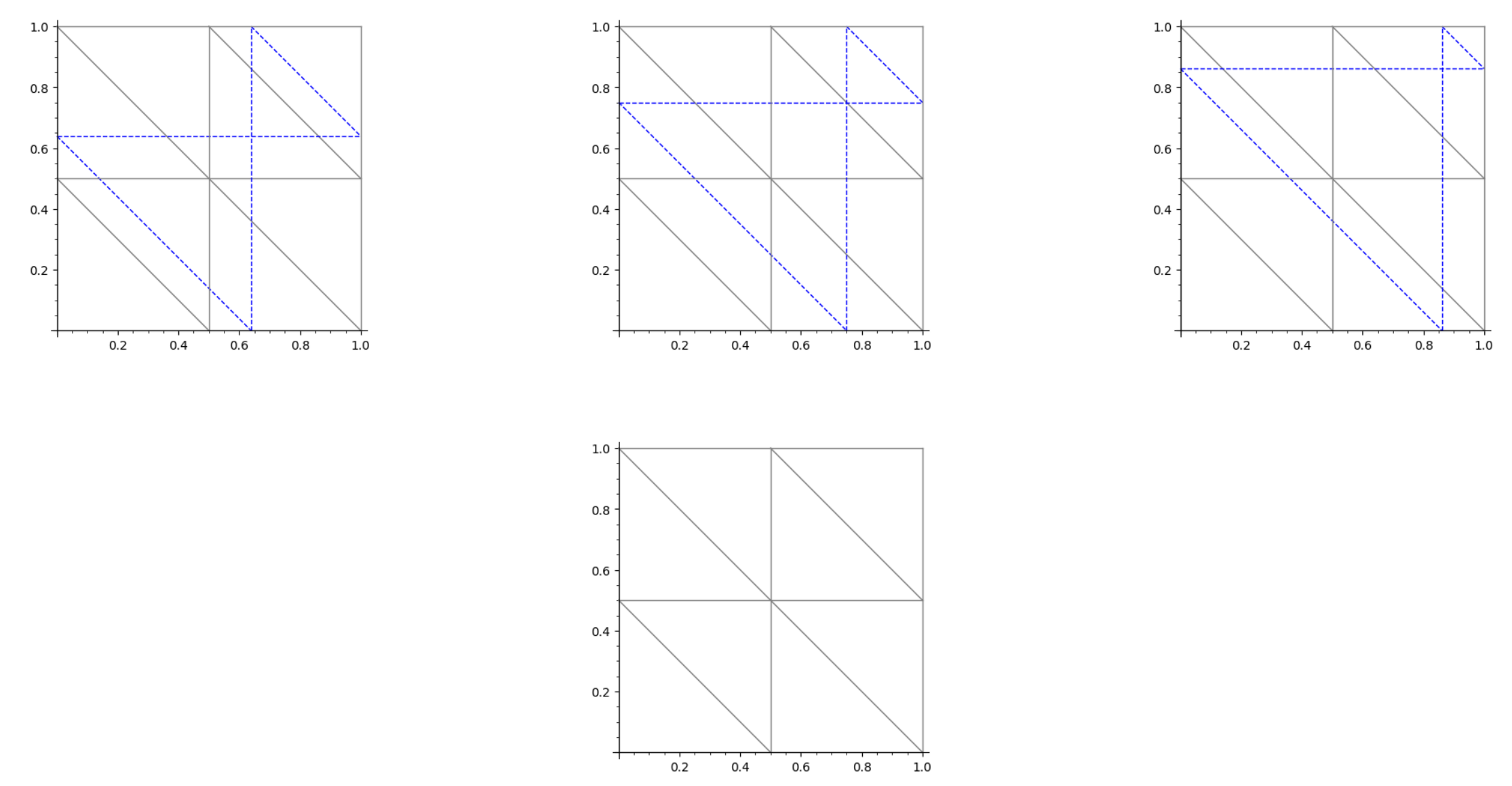
$$\Delta\pi_{(b,v)}(x, y) \bowtie_i 0 \iff \Delta\pi_{(\overline{b},\overline{v})}(x, y) \bowtie_i 0. \tag{8}$$

Why? Vertices identified though the isomorphism have the same combinatorial properties, so $\Delta\pi_{(b,v)}$ and $\Delta\pi_{(\overline{b},\overline{v})}$ evaluated at identified vertices have the same formula when written as parameters.

Observation 3. Clearing denominators and counting vertices gives a finite number of polynomials as in the statement. Apply this and previous observations to Theorem 5.

Corollary 11. *Suppose $n \geqslant 2$ is fixed. There exists a finite number of semialgebraic sets S_α for $\alpha \in \mathbb{A}_n$, $\alpha \in S_\alpha$, and $|\mathbb{A}_n| = O(n!)$ such that for all $p \in S_\alpha$, π_p is minimal and has at most n breakpoints and $\bigcup_{\alpha \in \mathbb{A}_n} \{\pi_p : p \in S_\alpha\} = \Pi^{\min}_{\leqslant n}$.*

Sketch of proof. (*Proposition*) Isomorphisms of face lattices induces an equivalence relation on $\Delta\mathcal{P}_b$. (*Lemma*) There are at most $O(n!)$ equivalence classes indexed by breakpoint sequences.



A breakpoint complex and potential different non isomorphic breakpoint complexes by adding an additional breakpoint.

Proposition 12. *Let $\overline{b}$ be a fixed breakpoint sequence of length $n \geqslant 2$. Then the set $V_{\overline{b}, \text{f index}} = \{(\overline{b}, v) \in \mathbb{R}^{2n} : \pi_{(\overline{b},v)}$ is minimal and $\gamma_{\text{f index}} = 1\}$ is a polyhedron.*

For all possible indices f index $\in \{1, \ldots, n-1\}$, $V_{\overline{b}, \text{f index}} \neq \varnothing$ as value parameters for gmi[cut generating function] with, for $n > 3$, virtual breakpoints is an element of $V_{\overline{b}, \text{f index}}$. For all breakpoint indices and for all f indices, construct the value polyhedron and pick a point in the value polyhedron to define representative elements. Use this to define $\mathbb{A}_n$. $(\supseteq)$ Let $\pi \in \Pi^{\text{min}}_{\leqslant n}$, bkpt $= \text{BKPT}(\pi)$, f index$(\pi) =$ f index, val $= \text{VAL}(\pi)$. There exists $\overline{b}$, a breakpoint index, $\Delta\mathcal{P}_{\overline{b}} \cong \Delta\mathcal{P}_{\text{bkpt}}$. There exists $(\overline{b}, \overline{v}) \in \mathbb{A}_n$ such that f index$(\pi_{(\overline{b},\overline{v})}) =$ f index. Then apply Theorem 10 to $(\overline{b}, \overline{v})$. Hence $(\text{bkpt}, \text{val}) \in \text{BSA}_{(\overline{b},\overline{v})} = S_{(\overline{b},\overline{v})}$ implying $\pi_{\text{bkpt,val})} \in \bigcup_{\alpha \in \mathbb{A}_n} \{\pi_p : p \in S_\alpha\}$. $(\subseteq)$ We take S_α to be the semialgebraic sets produced by Theorem 10 for the given index. By construction every function parameterized by a point in S_α is minimal, hence the inclusion holds.

Implementation: Precomputed (via HPC) index sets for $n = 2, \ldots, 7$ are available as an optional package for `cutgeneratingfunctionology` from the python package `MinimalFunctionCache`.

Future Development: Study smooth geometry of $\Pi^{\min}_{\leqslant k}$ and improve combinatorial cell description to solve full space cut optimization problem effectively.