Column generation and IP
From textbook to practice - Part I

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Motivation and basic info

Motivation

- Column generation (and branch-and-price) are an important tool in OR
- Coverage in textbooks (and courses) can be small/insufficient
- Present some of the basic concepts
- Present some recent works (mine and others), and issues not typically covered

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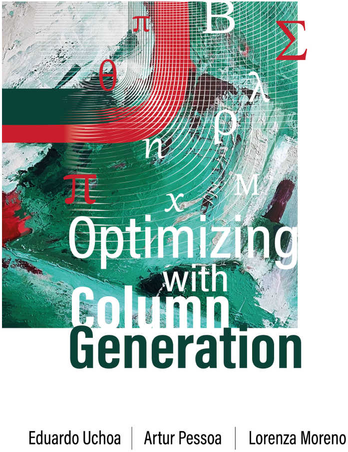
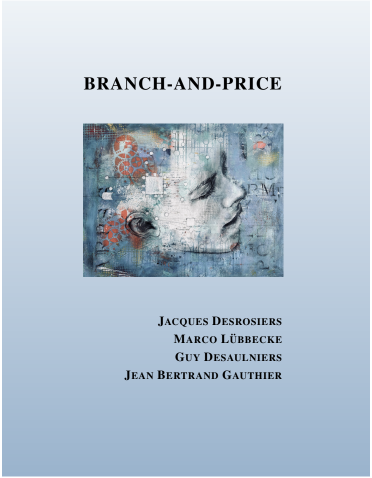
- Theme: "Proof is in the pudding"

What textbook?





Other useful sources

- Desrosiers and Lübbecke (2011)
- Barnhart et al. (1998)
- Desaulniers, Desrosiers, and Solomon (2005)

Some applications and high level comments

Some example applications:

- Routing applications
- Scheduling (airline, crew, nurse roster)
- Some CO problems: Knapsack variants, graph coloring, multicommodity flows, etc.
- Classification/clustering
- Decision rules
- ...

Some of the comments about advantages:

- Better GAPs (stronger relaxations)
- Fewer branching nodes
- More interpretable output
- Easier to handle “hard-to-model” constraints in classic compact IP

Some disadvantages

- More expensive branch-and-bound nodes
- Harder to implement

Color code

- Textbook
- Issues that are somewhat standard
- Research issues

The basics of CG

Column generation and LP

Suppose I solve the LP (RP) and its dual (RD):

$$\begin{array}{llllll} \min & 3x_1 & +2x_2 & +x_3 & & \\ \text{s.t.} & x_1 & +x_2 & & \geq 9 & \\ & x_1 & & & \geq 7 & \\ & & x_2 & +x_3 & = 6 & \\ & x_1, x_2, x_3 \geq 0 & & & & \end{array} \tag{RP}$$

$$\begin{array}{llllll} \max & 9\pi_1 & +7\pi_2 & +6\pi_3 & & \\ \text{s.t.} & \pi_1 & +\pi_2 & & \leq 3 & \\ & \pi_1 & & +\pi_3 & \leq 2 & \\ & & & +\pi_3 & \leq 1 & \\ & \pi_1, \pi_2 \geq 0 & & & & \end{array} \tag{RD}$$

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Opt $\bar{x} = (7, 2, 4)$, $\bar{\pi} = (1, 2, 1)$
Then:

- $\bar{x}$ is feasible for (RP)
- $\bar{\pi}$ is feasible for (RD)
- $\bar{x}, \bar{\pi}$ satisfy complementary slackness:
  - Either $\bar{x}_j = 0$ or corresponding dual constraint is tight at $\bar{\pi}$
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Column generation and LP

Suppose I want to solve the new LP (P) and its dual (D):

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Reuse $\bar{x} = (7, 2, 4, 0)$ and $\bar{\pi} = (1, 2, 1)$

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Then:

- $\bar{x}$ is feasible for (P) ✓
- $\bar{\pi}$ is feasible for (D) ? If yes, then $\bar{x}, \bar{\pi}$ optimal for (P) and (D).
- $\bar{x}, \bar{\pi}$ satisfy complementary slackness: ✓
  - Either $\bar{x}_j = 0$ or corresponding dual constraint is tight at $\bar{\pi}$ ✓
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Column generation and LP II: Generalizing

Suppose I want to solve (P) with $|\mathcal{R}|$ very large.

$$\begin{array}{ll} \min & \sum_{i \in \mathcal{R}} c_i \lambda_i \\ \text{s.t.} & \sum_{i \in \mathcal{R}} A_i \lambda_i \;\; ? \;\; b \\ & \lambda \geq 0 \end{array} \tag{P}$$

$$\begin{array}{ll} \max & b^T \pi \\ \text{s.t.} & A_i^T \pi \leq c_i, \forall i \in \mathcal{R} \\ & \pi \text{ satisfying appropriate sign contraints} \end{array} \tag{D}$$

Column generation and LP II: Generalizing

1. Pick $\mathcal{R}' \subseteq \mathcal{R}$ and solve (RP):

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2. Let $\bar{\lambda}, \bar{\pi}$ be the corresponding optimal solutions
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- Pricing problem: Does there exist i such that $A_i^T \bar{\pi} > c_i$?

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  Can be solved by finding element in $\mathcal{R}$ with smallest reduced cost.

Computational Issue: Stabilization

Stabilization

- "Vanilla" version of column generation typically suffers from tailing off
- "Bang-bang" behaviour of dual variables

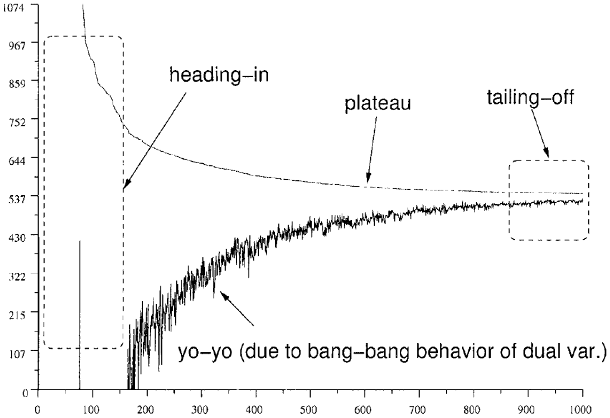

Figure: Taken from Vanderbeck, 2005

Stabilization II

Some options: (Marsten, Hogan, and Blankenship, 1975)

- Put a box around it
- Solve RMP/generate columns
- Possibly update the box and repeat

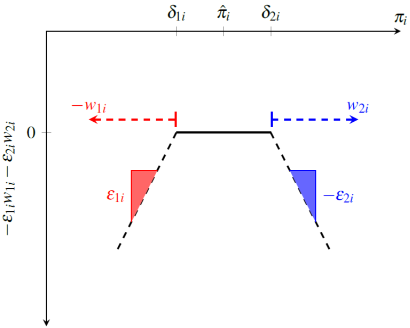

Figure: Taken from Desrosiers et al. (2024)

Stabilization II

Changes to (RD):

$$\begin{array}{ll} \max & b^T\pi - \sum_i (\varepsilon_{1i} w_{1i} + \varepsilon_{2i} w_{2i}) \\ \text{s.t.} & A_i^T \pi \leq d_i, \forall i \in \mathcal{R}' \\ & \pi_i \leq \hat{\pi}_i + \delta_{2i} + w_{2i} \\ & \pi_i \geq \hat{\pi}_i - \delta_{1i} - w_{1i} \\ & \pi \text{ satisfying appropriate sign contraints} \end{array} \tag{RD}$$

must be reflected back in the primal.

Stabilization III

Some options: (Pessoa et al., 2013)

- In out separation

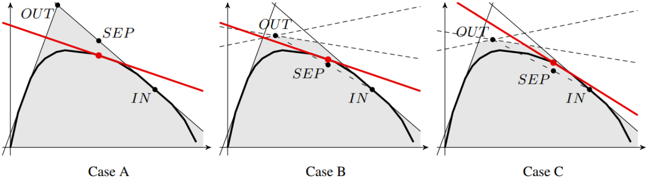

Figure: Taken from Pessoa et al. (2013)

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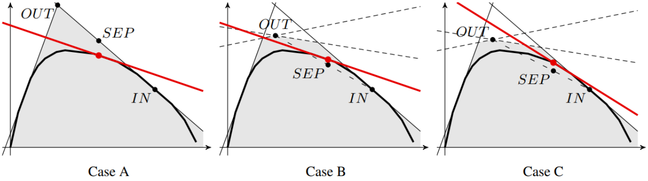

Figure: Taken from Pessoa et al. (2013)

1. Maintain inner point IN (known dual feasible) and outer point OUT (candidate dual that is being separated)
2. At iteration t, try to separate a point $SEP = \alpha^t IN + (1 - \alpha^t) OUT$
3. Update points

Recent stabilization work by Costa et al. (2022)

Dantzig-Wolfe decomposition and IP

Back to textbook: CG, Lagrangean relaxation and Dantzig-Wolfe

Consider the following problem:

$$\begin{aligned} z^* := \quad \min \quad & c^T x \\ \text{s.t.} \quad & x \in X \\ & Dx \geq f \end{aligned} \tag{P}$$

where X is a nonempty polytope for which we know how to optimize "easily" (i.e. fast in practice).

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Lagrangean dual:

$$z_{LAG} := \max_{\pi \geq 0} z(\pi) \tag{1}$$

where

$$z(\pi) = \begin{array}{ll} \min & c^T x + \pi^T (f - Dx) \\ \text{s.t.} & x \in X \end{array} \tag{2}$$

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Then

$$z(\pi) \leq z_{LAG} \leq z^*$$

CG, Lagrangean relaxation and Dantzig-Wolfe II

Let $\{v^i\}_{i\in\mathcal{R}}$ be the set of extreme points of X. Then
$x \in X \iff x$ conv. comb. of $\{v^i\}_{i\in\mathcal{R}}$
Then

$$
\begin{aligned}
z_{DW} := \quad \min \quad & \sum_{i\in\mathcal{R}} c^T v^i \lambda_i \\
\text{s.t.} \quad & \sum_{i\in\mathcal{R}} D v^i \lambda_i \geq f \\
& \sum_{i\in\mathcal{R}} \lambda_i = 1 \\
& \lambda \geq 0
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\tag{DW}
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$$z_{DW} := \begin{array}{ll} \min & \sum_{i \in \mathcal{R}} c^T v^i \lambda_i \\ \text{s.t.} & \sum_{i \in \mathcal{R}} D v^i \lambda_i \geq f \\ & \sum_{i \in \mathcal{R}} \lambda_i = 1 \\ & \lambda \geq 0 \end{array} \tag{DW}$$

Theorem

$$z_{DW} = z_{LAG} \leq z^*$$

What about IP?

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\begin{aligned}
z^* := \quad & \min \quad && c^T x \\
& \text{s.t.} && x \in X \\
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Theorem

$$z_{DW} = z_{LAG} \leq z^*$$

- If $X = \{x \in \mathbb{R}^n : Gx \geq h\}$, then $z_{DW} = z_{LAG} = z^*$
- If $X = \{x \in \mathbb{Z}^n : Gx \geq h\}$, then typically $z_{DW} = z_{LAG} < z^*$.

CG, Lagrangean relaxation and Dantzig-Wolfe II

Solving (DW). $\mathcal{R}' \subseteq \mathcal{R}$

$$
\begin{aligned}
z_{DWR} := \quad \min \quad & \sum_{i \in \mathcal{R}'} c^T v^i \lambda_i \\
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Pricing problem: Given $(\bar{\pi}, \bar{\pi}_o)$ optimal for the dual of (DWR), find

$$z_{PR} = \min_{i \in \mathcal{R}} c^T v^i - \bar{\pi}^T D v^i - \bar{\pi}_o = \begin{array}{ll} \min & (c^T - \bar{\pi}^T D)x - \bar{\pi}_o \\ \text{s.t.} & x \in X \end{array} \tag{PRIC}$$

Getting a lower (dual) bound

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Sometimes mentioned as a drawback of CG:
Does not give a lower bound until the last iteration.

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Theorem (Taken from Uchoa, Pessoa, and Moreno, 2024)

$z_{DWR} + z_{PR} \leq z_{DW} = z_{LAG} \leq z^*$

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Then $z_{PR} \leq 0$. Why?

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There exists $i \in \mathcal{R}'$ such that $\bar{\lambda}_i > 0$.

By complementary slackness, corresponding dual constraint is tight (i.e. reduced cost is 0). But since $\mathcal{R}' \subseteq \mathcal{R}$, then $z_{PR} \leq 0$.

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$-z_{PR}$ is maximum constraint violation of (D)

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$-z_{PR}$ is maximum constraint violation of (D)

$(\bar{\pi}, \bar{\pi}_o + z_{PR})$ is a feasible solution to (D) of value $z_{RM} + z_{PR}$

DW decomposition for block-structure

DW decomposition

Dantzig-Wolfe (DW) decomposition often described for problems of the form:

$$
\begin{array}{llllll}
\min & (c^1)^T x^1 & +(c^2)^T x^2 & +\dots & (c^K)^T x^K & \\
\text{s.t.} & D^1 x^1 & +D^2 x^2 & +\dots & +D^K x^K \geq f & \\
 & x^1 \in X^1 & & & & \\
 & & x^2 \in X^2 & & & \text{(BL)} \\
 & & & \dots & & \\
 & & & & x^K \in x^K &
\end{array}
$$

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Removing linking constraints $\rightarrow$ Problem is decomposable.

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Same idea applies: Write each x^i as convex combination of extreme points of X^i.

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Removing linking constraints $\rightarrow$ Problem is decomposable.

Same idea applies: Write each x^i as convex combination of extreme points of X^i.

If all blocks are identical (same costs, feasible region):

$$\begin{array}{ll}
\min & \sum_{i \in \mathcal{R}} c^T v^i \lambda_i \\
\text{s.t.} & \sum_{i \in \mathcal{R}} D v^i \lambda_i \geq f \\
& \sum_{i \in \mathcal{R}} \lambda_i = K \\
& \lambda \geq 0
\end{array} \qquad \text{(DWK)}$$

Branching

Branch-and-price

$$z_{IP}^* := \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Dx \geq f \\ & Gx \geq h \\ & x \in \mathbb{Z}^n \end{array} \tag{IP}$$

We can reformulate (IP) as:

$$z_{IP}^* = \begin{array}{ll} \min & \sum_{i \in \mathcal{R}} c^T v^i \lambda_i \\ \text{s.t.} & \sum_{i \in \mathcal{R}} D v^i \lambda_i \geq f \\ & \sum_{i \in \mathcal{R}} \lambda_i = 1 \\ & \lambda_i \in \{0, 1\}, \forall i \in \mathcal{R} \end{array} \tag{DW-IP}$$

with $X = conv\{x \in \mathbb{Z}^n : Gx \geq h\}$

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Branch-and-price: Solve (DW-IP) - each LP in a branch-and-bound node is solved via CG

Branching

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\begin{aligned}
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& \lambda_i \in \{0,1\}, \forall i \in \mathcal{R}
\end{aligned}
\tag{DW-IP}
$$

- Standard branching: Set a variable $\lambda_i = 0$ or $\lambda_i = 1$
- Branch with $\lambda_i = 1$ is immediately solved, branch with $\lambda_i = 0$ almost does not change $\rightarrow$ imbalanced BB tree
- More importantly, consider node with $\lambda_i = 0$.
  - Pricing $\rightarrow$ Find smallest reduced cost element of $\mathcal{R} \setminus \{i\}$
  - Like finding k-smallest reduced cost element: Becomes increasingly harder to solve.
- Ryan and Foster (1981) propose a better branching
- F. et al. (2006): See next.

Issue: Dealing with new inequalities

Robust branch-price-and-cut (BPC) - (F. et al., 2006)

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Dx \geq f \\
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- Idea: Combine the two formulations

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & x & - \sum_{i \in \mathcal{R}} v^i \lambda_i & = 0 \\ & Dx & & \geq f \\ & & \sum_{i \in \mathcal{R}} \lambda_i & = 1 \\ & x \in \mathbb{Z}^n & & \\ & & \lambda_i \in [0,1], \forall i \in \mathcal{R} & \end{array} \tag{BPC}$$

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- So adding cuts/branching on x will not affect pricing $\rightarrow$ Robust BPC

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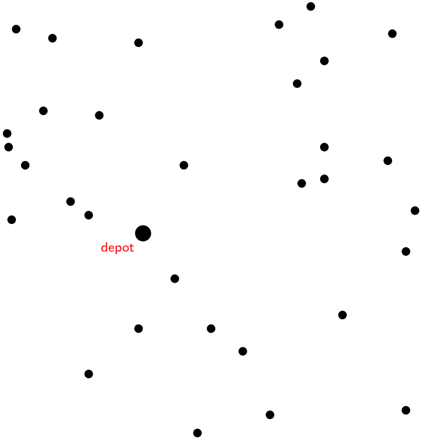
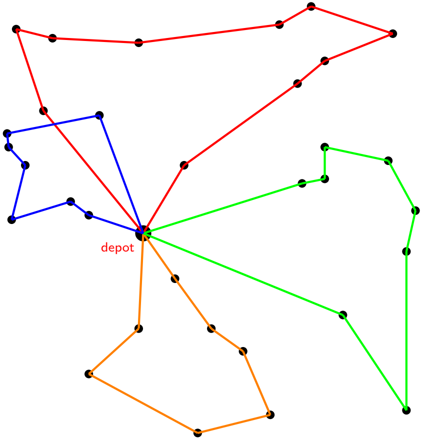
- So adding cuts/branching on x will not affect pricing $\rightarrow$ Robust BPC
- Uchoa, Pessoa, and Moreno, 2024: Better to project out x variables

The capacitated vehicle routing problem (CVRP)



- $G = (V, E)$
- $V = \{0\} \cup V_+$
- Edge lengths $\ell_e, \ e \in E$
- K vehicles, capacity C
- Client demands $d_i, \forall i \in V_+$.
- Let S_j be the set of clients served by route j.
  Then $d(S_j) := \sum_{u \in S_j} d_u \leq C$
- Goal: Find minimum cost set of K routes that start/end at depot, serves all customers and respect capacity constraint

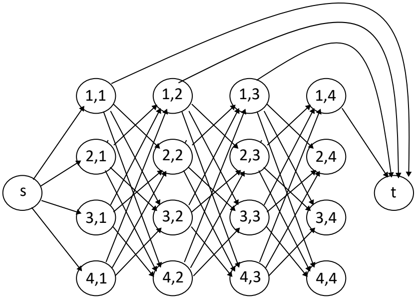
The capacitated vehicle routing problem (CVRP)



- $G = (V, E)$
- $V = \{0\} \cup V_+$
- Edge lengths $\ell_e,\ e \in E$
- K vehicles, capacity C
- Client demands $d_i, \forall i \in V_+$.
- Let S_j be the set of clients served by route j.
  Then $d(S_j) := \sum_{u \in S_j} d_u \leq C$
- Goal: Find minimum cost set of K routes that start/end at depot, serves all customers and respect capacity constraint

Pricing

- q-routes: Walks that start/end at depot and satisfy capacity
- Pricing: Finding minimum cost q-route
- Shortest path in a "state-space" graph
- States are (v, δ).
    - v: Last visited client
    - δ: Total accumulated demand
- Minimum cost q-route found in pseudopolynomial time.
- If we want to forbid cycles: strongly NP-hard



Pricing

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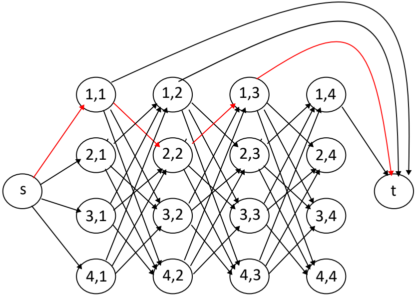

Corresponding q-route: (0,1,2,1,0)

CVRP formulation (F. et al., 2006)

$$
\begin{array}{llllll}
\min & \sum_{e\in E} \ell_e x_e & & & & \\
\text{s.t.} & x_e & -\sum_{r\in\mathcal{R}} \textsc{count}(e,r)\cdot\lambda_r & = 0, & \forall e \in E, \\
& \sum_{e\in\delta(0)} x_e & & = 2K, & \\
& \sum_{e\in\delta(v)} x_e & & = 2, & \forall v \in V_+ \\
& \sum_{e\in\delta(S)} x_e & & \geq \left\lceil \frac{d(S)}{C} \right\rceil & \forall \emptyset \subsetneq S \subseteq V_+ \\
& & \lambda \geq 0 & & \\
& x_e \in \{0,1\} & & & \forall e \in E \setminus \delta(0) \\
& x_e \in \{0,1,2\} & & & \forall e \in \delta(0).
\end{array}
\tag{SP}
$$

- $\mathcal{R}$: Set of q-routes
- $\textsc{count}(e,r)$: Number of times r goes through edge e
- Reduced cost of λ_r: $-\sum_{e\in E} \textsc{count}(e,r)\pi_e$

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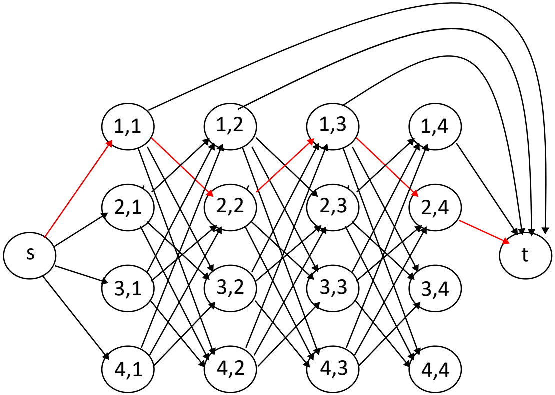
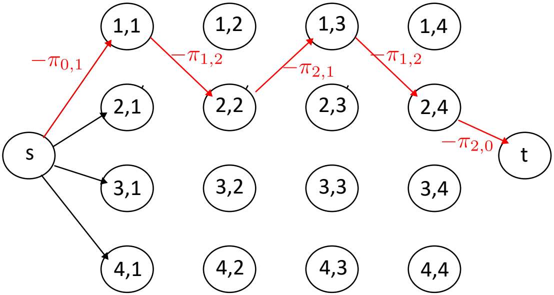
Pricing

- Finding minimum reduced cost q-route



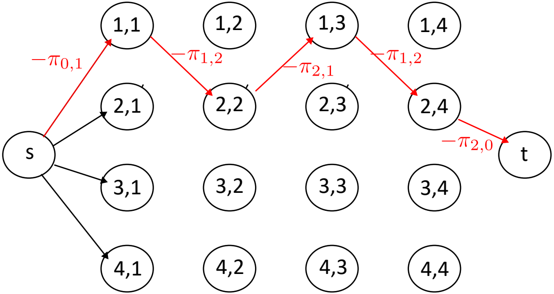
Pricing

- Finding minimum reduced cost q-route



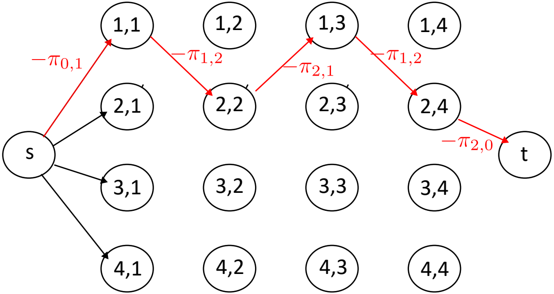
Pricing

- Finding minimum reduced cost q-route
- Edge 12 appears 3 times, so $\textsc{count}(12, r) = 3$



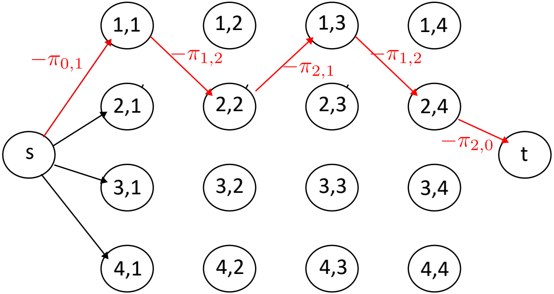
Pricing

- Finding minimum reduced cost *q*-route
- Edge 12 appears 3 times, so $\text{COUNT}(12, r) = 3$
- Reduced cost of given *q*-route is $-\pi_{0,1} - 3\pi_{1,2} - \pi_{2,0}$



Pricing

- Finding minimum reduced cost q-route
- Edge 12 appears 3 times, so $\textsc{count}(12, r) = 3$
- Reduced cost of given q-route is $-\pi_{0,1} - 3\pi_{1,2} - \pi_{2,0}$
- Adding inequalities on x don't change way reduced cost is calculated (may change values of π)



Set partitioning formulation (Christofides, Mingozzi, and Toth, 1981)

$$
\begin{array}{lll}
\min & \sum_{r\in\mathcal{R}} c_r \cdot \lambda_r & \\
\text{s.t.} & \sum_{r\in\mathcal{R}} \text{COUNT}(v,r)\cdot\lambda_r = 1, & \forall v \in V_+, \\
& \sum_{r\in\mathcal{R}} \lambda_r = K, & \\
& \lambda_r \in \{0,1\}, & \forall r \in \mathcal{R}.
\end{array}
\tag{SP}
$$

- $\mathcal{R}$: Subset of possible closed walks $r = (0, v_1, \dots, v_k, 0)$, $v_i \in V_+$: r respects capacity constraint
- $\text{COUNT}(v, P)$: Number of times r goes through v
- This is a DW reformulation of the problem
- It is also the projection of F. et al. (2006) without "subtour-like" constraints

Time for a break (Insert dad joke here)

Question

How do you get in touch with a roman architect?

Time for a break (Insert dad joke here)

Question

How do you get in touch with a roman architect?

You column.

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