Introduction to cutting planes for mixed integer linear (nonlinear) programs

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Section 1

Introduction
Cuts: obtaining better dual bounds

Mixed integer linear program

\[
z^{OPT} := \max \quad c^\top x \\
\text{s.t.} \quad Ax \leq b \quad \text{(convex constraints)}
\]

\[
x \in \mathbb{Z}^{m_1} \times \mathbb{R}^{n_2}. \quad \text{(non-convex constraints)}
\]
Cuts: obtaining better dual bounds

Mixed integer linear program

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1. Feasible solution \( \hat{x} \): \( (c^T \hat{x}) \) provides a lower bound on \( z^{\text{OPT}} \).
Cuts: obtaining better dual bounds

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Ax \leq b \quad \text{(convex constraints)} \\
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1. Feasible solution \( \hat{x} \): \( (c^\top \hat{x}) \) provides a lower bound on \( z^{\text{OPT}} \).

2. Solving convex (LP) relaxation gives (standard) dual (upper) bound \( (z^{\text{LP}}) \).

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Ax \leq b \quad \text{(convex constraints)} \\
x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad \text{(non-convex constraints)}
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\]

\[ z^{\text{LP}} \geq z^{\text{OPT}} \geq c^\top \hat{x} \]
Cuts: obtaining better dual bounds

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3. Perfect dual bound \( (z^{OPT}) \) comes from solving convex hull of feasible region.

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z^{OPT} = \max \ c^T x \\
\text{s.t.} \quad x \in \text{conv}(\{x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} | Ax \leq b\}) \quad \text{(convex hull)}
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Cuts: obtaining better dual bounds

Mixed integer linear program

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4. Improving LP dual bound by adding cutting-planes.

\[
\begin{align*}
z^{LP+CUTS} & := \max c^T x \\
s.t. & \ A x \leq b \quad \text{(convex constraints)} \\
& \ \tilde{A} x \leq \tilde{b} \quad \text{(valid for convex hull – Cuts)}
\end{align*}
\]
Mixed integer linear program

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\[ z^{LP} \geq z^{LP+CUTS} \geq z^{OPT} \geq c^\top \hat{x} \]
An integer program: feasible region
An integer program: objective function

\[ c^T x \]
An integer program: optimal solution
An integer program: dual bound from LP relaxation

Optimal LP Solution Gives Upper bound

$\mathbf{c}^T \mathbf{x}$
An integer program: perfect dual bound from convex hull

Upper Bound = Lower Bound
An integer program: improved dual bound using cutting-plane(s)
Why linear inequalities is a reasonable choice:  
Fundamental theorem of integer programming

**Theorem ([Meyer (1974)])**

Let $S := \{x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \mid Ax \leq b\}$. If $A$ and $b$ is rational, then $\text{conv}(S)$ is a rational polyhedron.
Why linear inequalities is a reasonable choice: 
Fundamental theorem of integer programming

**Theorem ([Meyer (1974)])**

Let \( S := \{ x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} | Ax \leq b \} \). If \( A \) and \( b \) is rational, then \( \text{conv}(S) \) is a rational polyhedron.

- Also adding linear cutting-plane, means we need to only solve modified LPs with dual simplex.
- Generalization of the above result for integer points in general convex set: [D., Morán (2013)]
How to generate cutting-planes?

- **Geometric ideas**: Split Disjunctive cuts, Chvátal-Gomory Cuts, maximal lattice-free cuts.
- **Subadditive inequalities**: Gomory mixed integer cut.
- **Cuts using algebraic properties**: Extended formulations.
- **Cut from structured relaxations**: Boolean quadric polytope, Clique cuts, Mixed integer rounding inequalities, Lifted cover, Flow cover, Mixing inequalities, . . .
- **Lifting**: A technique to generate, rotate and strengthen inequalities. (Not covering this technique here)
- . . .
Section 2

Geometric Ideas
2.1

Split disjunctive cuts
Split disjunctive cuts

[Balas (1979)][Cook, Kannan, Schrijver (1990)]

Let $P \subseteq \mathbb{R}^n$ be a set and we are interested in obtaining valid inequality for $P \cap \mathbb{Z}^n$. 
Split disjunctive cuts

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Let $P \subseteq \mathbb{R}^n$ be a set and we are interested in obtaining valid inequality for $P \cap \mathbb{Z}^n$.

Let $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$.

Since

$$\mathbb{Z}^n \cap \{x \in \mathbb{R}^n \mid \pi_0 < \pi^T x < \pi_0 + 1\} = \emptyset.$$
Split disjunctive cuts

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Split disjunctive set

If $\alpha^\top x \leq \beta$ is valid for:

- $P \cap \{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0\}$, and
- $P \cap \{x \in \mathbb{R}^n \mid \pi^\top x \geq \pi_0 + 1\}$, then $\alpha^\top x \leq \beta$,

is valid inequality for

\[
P^{\pi, \pi_0} := \text{conv} \left( \left( P \cap \{x \in \mathbb{R}^n \mid \pi_0 \geq \pi^\top x \} \right) \cup \left( P \cap \{x \in \mathbb{R}^n \mid \pi^\top x \geq \pi_0 + 1\} \right) \right)
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and therefore also for: $P \cap \mathbb{Z}^n$. 
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- Let $P \subseteq \mathbb{R}^n$ be a set and we are interested in obtaining valid inequality for $P \cap \mathbb{Z}^n$.
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Special-case: Chvátal-Gomory Cuts

[Gomory (1958)]

If (WLOG) \( P \cap \{ x \in \mathbb{R}^n \mid \pi^T x \geq \pi_0 + 1 \} = \emptyset \), then \( \pi^T x \leq \pi_0 \) is a valid inequality for \( P \cap \mathbb{Z}^n \).

Follow-up work: [Schrijver (1980)], [Dadush, D., Vielma (2014)], [Cornuéjols, Lee (2018)]
Special-case: Chvátal-Gomory Cuts

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Follow-up work: [Schrijver (1980)], [Dadush, D., Vielma (2014)], [Cornuéjols, Lee (2018)]
Main take aways

▶ Given a set $P$, find a set $T$ such that

$$\text{int}(T) \cap \mathbb{Z}^n = \emptyset.$$
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(in fact it is enough to satisfy $P \cap \text{int}(T) \cap \mathbb{Z}^n = \emptyset$).
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- Given a set $P$, find a set $lattice-free set T$ such that

$$\text{int}(T) \cap \mathbb{Z}^n = \emptyset.$$ 

(in fact it is enough to satisfy $P \cap \text{int}(T) \cap \mathbb{Z}^n = \emptyset$).

- Find an inequality valid $\alpha^T x \leq \beta$, valid for $P \setminus \text{int}(T)$. 

- What type of lattice-free set $T$ considered?
  - non-convex?
  - convex?
  - polyhedral?

- How is the valid inequality found?
  - Valid inequality for $\text{conv}(P \setminus \text{int}(T))$.
  - Closed-form "formula"?
Main take aways

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1.2
Generalizations of split disjunctive cuts
Types of lattice-free $T$ sets I: non-convex

- **Asymmetric** [Dash, D., Günlük (2012)].

  Divides the feasible region into smaller polyhedral sets whose union contains all the integer points.
Types of lattice-free $T$ sets I: non-convex

- *Asymmetric* [Dash, D., Günlük (2012)].

- *Union of split disjunctions* [Li, Richard (2008)], [Dash et al. (2013)], [Dash, Günlük, Morán (2013)]

Divides the feasible region into smaller polyhedral sets whose union contains all the integer points.
Types of lattice-free $T$ sets II: convex

[Lovász (1989)]

- $T$ is a convex set that does not contain integers in its interior:
  Lattice-free convex set.
Types of lattice-free $T$ sets II: convex

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- $T$ is a convex set that does not contain integers in its interior: Lattice-free convex set.

- Lattice-free cuts can give the convex hull of the mixed-integer feasible solutions. Picture proof:
Types of lattice-free $T$ sets II: convex

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Maximal lattice-free convex set

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**Definition (Maximal Lattice-free convex set)**

We say $T \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set if $T' \subseteq \mathbb{R}^n$ is a lattice-free convex set and $T' \supseteq T$, implies $T' = T$. 
Maximal lattice-free convex set

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**Theorem ([Lovász (1989)], [Basu, Conforti, Cornuéjols, Conforti (2010)])**

All maximal lattice-free convex sets are polyhedral. Moreover, a full-dimension lattice-free convex set is maximal iff it is a lattice-free polyhedron with integer point in the relative interior of its facets.
Maximal lattice-free convex set
Generalization of maximal lattice-free sets
Generalization of maximal lattice-free sets

$S = P \cap \mathbb{Z}^2$

$S$-Free convex set
Generalization of maximal lattice-free sets

\[ S = P \cap \mathbb{Z}^2 \]

S-Free convex set
Definition (Maximal S-free convex set; [Johnson (1983)], [D., Wolsey (2010)])

Let $S = P \cap \mathbb{Z}^n$, where $P$ is a convex set. We say:

- $T$ is a convex $S$-free set, if $\text{int}(T) \cap S = \emptyset$.

- $T \subseteq \mathbb{R}^n$ is a maximal $S$-free convex set if $T' \subseteq \mathbb{R}^n$ is a $S$-free convex set and $T' \supseteq T$, implies $T' = T$. 
Generalization of maximal lattice-free sets

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Theorem ([D., Morán (2011)])

All maximal $S$-free convex sets are polyhedral.
Let maximal lattice-free (or S-free) set be $T := \{x \in \mathbb{R}^n \mid (g^i)^\top x \geq h^i \mid i \in [m]\}$. 

If $\alpha^\top x \leq \beta$ is valid for the disjunction $m \_i = 1 \_P \cap \{z\mid (g^i)^\top x \leq h^i \mid i \in [m]\}$, then $\alpha^\top x \leq \beta$ is a valid inequality for $P \cap \mathbb{Z}^n$. 

One approach to find inequality $\alpha^\top x \leq \beta$ to separate $x^*$. See [Balas, Perregaard: (2003)] for a method to generate cuts for split disjunctions with just one copy of variables (instead of two copies).
Polyhedrality of maximal lattice-free sets is useful

- Let maximal lattice-free (or S-free) set be
  \[ T := \{ x \in \mathbb{R}^n \mid (g^i)^\top x \geq h^i \ i \in [m] \}. \]
- If \( \alpha^\top x \leq \beta \) is valid for the disjunction:
  \[
  \bigvee_{i=1}^{m} P \cap \left\{ x \in \mathbb{R}^n \mid (g^i)^\top x \leq h^i \right\},
  \]
  complement of a facet of \( T \)
  then \( \alpha^\top x \leq \beta \) is a valid inequality for \( P \cap \mathbb{Z}^n \).
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- Let maximal lattice-free (or S-free) set be
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- One approach to find inequality \( \alpha^T x \leq \beta \) to separate \( x^* \):

  \[
  \max_{\alpha, \beta} \quad \alpha^T x^* - \beta \]
  
  s.t. \( \alpha x \leq \beta \) is valid for \( \left( P \cap \{ x \in \mathbb{R}^n \mid (g^i)^T x \leq h^i \} \right) \forall i \in [m] \)
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One approach to find inequality \( \alpha^\top x \leq \beta \) to separate \( x^* \): Use Farkas Lemma:

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\begin{align*}
\max_{\alpha,\beta,\lambda,\mu} & \quad \alpha^\top x^* - \beta \\
\text{s.t.} & \quad \alpha^\top = (\lambda^i)^\top A + \mu^i \cdot (g^i)^\top \forall i \in [m] \\
& \quad \beta \geq (\lambda^i)^\top b + \mu^i \cdot h^i \forall i \in [m] \\
& \quad \lambda^i \geq 0, \mu^i \geq 0 \forall i \in [m] \quad \text{Cone}
\end{align*}
\]

Normalization constraint: either bound \( \alpha \) or \( \beta \).
Polyhedrality of maximal lattice-free sets is useful

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Final comments

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- This is very general paradigm: See, for example,
  - Disjunctive ideas to get convex hull of QCQPs: [Tawarmalani, Richard, Chung (2010)], [D., Santana (2020)]
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  - **Disjunctive ideas** to get convex hull of QCQPs: [Tawarmalani, Richard, Chung (2010)], [D., Santana (2020)]

- The real challenge is how to select the lattice-free set.
Section 3

Subadditive cutting-planes
A simple observation

- **Subadditive function**: A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is called subadditive if:
  \[ f(u) + f(v) \geq f(u + v) \text{ for all } u, v \in \mathbb{R}^m. \]

- **Non-decreasing function**: A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is called non-decreasing if:
  \[ f(u) \leq f(v) \text{ for all } u \leq v. \]
A simple observation

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**Theorem ([Gomory, Johnson (1972ab)], [Jeroslow (1978)],[Jeroslow (1979)], [Blair, Jeroslow (1982)])**

Let $S := \left\{ x \in \mathbb{R}^n_+ \left| \sum_{j=1}^n A^j x_j \geq b, \ x \in \mathbb{Z}^n \right. \right\}$, where $A^j \in \mathbb{R}^m$ for $j \in [n]$ and $b \in \mathbb{R}^m$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a subadditive function, non-decreasing, such that $f(0) = 0$, then

\[ \sum_{j=1}^n f(A^j) x_j \geq f(b), \]

is a valid inequality for $S$. 

\[ \square \]
Example of subadditive function

Consider the following set:

\[ S := \left\{ x \in \mathbb{Z}_+^3 \mid \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} x_3 \geq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]
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Consider the function \( f : \mathbb{R}^3 \to \mathbb{R} : \)

\[ f(u) = \lceil 0.5 \cdot (u_1 + u_2 + u_3) \rceil \]

This function is

- subadditive,
- non-decreasing,
- and \( f(0) = 0. \)
Example of subadditive function

Consider the following set:

\[ S := \left\{ x \in \mathbb{Z}_+^3 \mid \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x_3 \geq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

Consider the function \( f : \mathbb{R}^3 \to \mathbb{R} : \)

\[ f(u) = \lceil 0.5 \cdot (u_1 + u_2 + u_3) \rceil \]

This function is

- subadditive,
- non-decreasing,
- and \( f(0) = 0. \)

So we have the following valid inequality for \( S : \)

\[ f \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right) x_1 + f \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) x_2 + f \left( \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right) x_3 \geq f \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \]
Example of subadditive function

Consider the following set:

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Consider the function \( f : \mathbb{R}^3 \rightarrow \mathbb{R} : \)

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This function is

- subadditive,
- non-decreasing,
- and \( f(0) = 0 \).

Equivalently:

\[ x_1 + x_2 + x_3 \geq 2, \]

which is a facet-defining inequity for \( \text{conv}(S) \).
Mixed integer version

Theorem ([Gomory, Johnson (1972ab)])

Consider the set:

\[ S := \left\{ x \in \mathbb{R}^n_+ \left| \sum_{j=1}^{n} A^i x_j \geq b, \ x_j \in \mathbb{Z} \ j \in I \right. \right\}, \]

where \( A^i \in \mathbb{R}^m \) for \( j \in [n] \) and \( b \in \mathbb{R}^m \).

- Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a subadditive function, non-decreasing, such that \( f(0) = 0 \), and
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- Let \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) be a subadditive function, non-decreasing, such that \( f(0) = 0 \), and

- Let \( \bar{f}(u) := \limsup_{\epsilon \rightarrow 0^+} \left( \frac{f(u \epsilon)}{\epsilon} \right) \). Let \( \bar{f}(A^j) < \infty \) for all \( A^j \in [n] \setminus I \),

then

\[ \sum_{j \in I} f(A^j)x_j + \sum_{j \in [n]\setminus I} \bar{f}(A^j)x_j \geq f(b), \]

is a valid inequality for \( S \).
Theorem ([Gomory, Johnson (1972)])

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\[
\sum_{j \in I} f(A^i) x_j + \sum_{j \in [n] \setminus I} \bar{f}(A^i) x_j \geq f(b)
\]
A very very special sub-additive function: Gomory mixed integer cut (GMIC) [Gomory, Johnson (1972ab)]

\[ S := \left\{ (x, y) \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \mid \sum_{j=1}^{n_1} a_j x_j + \sum_{i=1}^{n_2} d_i y_i = b \right\}. \]
A very very special sub-additive function: Gomory mixed integer cut (GMIC)

[Goemory, Johnson (1972a,b)]

- \( S := \{(x, y) \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \mid \sum_{j=1}^{n_1} a_j x_j + \sum_{i=1}^{n_2} d_i y_i = b\} \).
- Let \( \text{frc}(a) = a - \lfloor a \rfloor \).
- \( f^{\text{GMIC}}(u) = \min \left\{ \frac{\text{frc}(u)}{\text{frc}(b)}, \frac{1 - \text{frc}(u)}{1 - \text{frc}(b)} \right\}, \quad \overline{f^{\text{GMIC}}}(u) = \begin{cases} \frac{u}{\text{frc}(b)} & u \geq 0 \\ \frac{(-u)}{(1 - \text{frc}(b))} & u \leq 0 \end{cases} \)
A very very special sub-additive function: Gomory mixed integer cut (GMIC)

[Goemory, Johnson (1972ab)]

\[ S := \{(x, y) \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \mid \sum_{j=1}^{n_1} a_j x_j + \sum_{i=1}^{n_2} d_i y_i = b \} . \]

\[ \text{Let } frc(a) = a - \lfloor a \rfloor . \]

\[ f^{GMIC}(u) = \min \left\{ \frac{frc(u)}{frc(b)}, \frac{1 - frc(u)}{1 - frc(b)} \right\} , \quad \overline{f^{GMIC}}(u) = \begin{cases} \frac{u}{frc(b)} & u \geq 0 \\ \frac{-u}{1 - frc(b)} & u \leq 0 \end{cases} \]

\[ \text{Gomory-mixed integer cut:} \]

\[ \sum_{j \in [n_1], frc(a_j) \leq frc(b)} \frac{frc(a_j)}{frc(b)} x_j + \sum_{j \in [n_1], frc(a_j) \geq frc(b)} \frac{1 - frc(a_j)}{1 - frc(b)} x_j \]

\[ \sum_{i \in [n_2], d_i \geq 0} \frac{d_i}{frc(b)} + \sum_{i \in [n_2], d_i \leq 0} \frac{-d_i}{1 - frc(b)} \geq 1. \]
A zoo of subadditive functions

![GMIC ◁ GMIC](image1)
![Two Slope ◁ GMIC](image2)
![Three Slope ◁ GMIC](image3)

![GMIC ◁ Two Slope](image4)
![Two Slope ◁ Two Slope](image5)
![Three Slope ◁ Two Slope](image6)

![GMIC ◁ Three Slope](image7)
![Two Slope ◁ Three Slope](image8)
![Three Slope ◁ Three Slope](image9)
A zoo of subadditive functions


Automatic search of these functions: [Köppe, Zhou (2016)] and follow up work.

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How good are these “subadditive cuts”?


Consider the set:

$$S := \left\{ x \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} A^j x_j \geq b, \ x_j \in \mathbb{Z} \ j \in I \right\},$$

where all the data is rational. Then the convex hull of $S$ can be obtained using inequalities generated by non-decreasing, subadditive functions (with $f(0) = 0$).
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\]

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Only a particular type of subadditive functions called as Chvátal functions are necessary for the above result: [Blair, Jeroslow (1982)], [Basu, Martin, Ryan, Wang (2019)]
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where all the data is rational. Then the convex hull of \( S \) can be obtained using inequalities generated by non-decreasing, subadditive functions (with \( f(0) = 0 \)).

**Theorem (Wolsey [1981])**

Consider the set:

\[ S(b) := \left\{ x \in \mathbb{Z}_+^n \left| \sum_{j=1}^{n} A^i x_j = b, \right. \right\}. \]

For \( A \) fixed, there is a finite list of subadditive functions that give the convex hull of \( S(b) \) for all \( b \).
How good are these “subadditive cuts”?


Consider the set:

\[ S := \left\{ x \in \mathbb{R}^n_+ \left| \sum_{j=1}^n A^i x_j \geq b, \ x_j \in \mathbb{Z} \ j \in I \right. \right\}, \]

where all the data is rational. Then the convex hull of \( S \) can be obtained using inequalities generated by non-decreasing, subadditive functions (with \( f(0) = 0 \)).

Theorem ([D., Morán, Vielma (2012)])

Consider the set:

\[ S := \left\{ x \in \mathbb{R}^n_+ \left| \sum_{j=1}^n A^i x_j \geq_K b, \ x_j \in \mathbb{Z} \ j \in I \right. \right\}, \]

where \( K \) is a proper cone and there exists a strictly feasible solution \( \hat{x} \). Then the convex hull of \( S \) can be obtained using inequalities generated by non-decreasing (appropriately defined wrt \( K \)), subadditive functions (with \( f(0) = 0 \)).

Follow-up: [Kocuk, Morán (2019)]
Any connection between maximal lattice-free convex cuts and subadditive cuts?

- We can obtain the convex hull using maximal lattice-free convex cuts and also subadditive cuts — is there a connection?
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- We can obtain the convex hull using **maximal lattice-free convex cuts** and also **subadditive cuts** — is there a connection? **YES!**
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One relationship via “intersection cuts” viewpoint of the lattice-free convex cuts for the set, \( \{ x \in \mathbb{Z}^m, z \in \mathbb{Z}^{n_1}_+, y \in \mathbb{R}^{n_2}_+, | x = b + Az + Gy \} \). Cuts in \((y, z)\)-space (Sketch):

**Subadditive function \( f \)**

Subadditive and sublinear function
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Subadditive function \((f)\)

\[
\downarrow \quad \left( \text{Slope of } f: \lim_{\epsilon \to 0^+} \frac{f(u\epsilon)}{\epsilon} \right)
\]
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\text{Subadditive function } (f) \\
\downarrow \\
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\[
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\]

\( \bar{f} \) Subadditive and sublinear function

\[
( T = \{ x | \bar{f}(x - v) \leq 1 \} )^a
\]
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\[
\text{Subadditive function } (f) \\
\downarrow \\
\text{Slope of } f: \lim_{\epsilon \to 0^+} \frac{f(u\epsilon)}{\epsilon} \\
\downarrow \\
\tilde{f} \quad \text{Subadditive and sublinear function} \\
\downarrow \\
(T = \{x|\tilde{f}(x - v) \leq 1\})^a
\]

A lattice-free convex set \( T \) around fractional point \( v \)

From \( \tilde{f} \) to lattice-free convex set: [Borozan Cornuéjols (2009)], [Conforti et al.(2015)]

\(^a\text{With proper scaling of } \tilde{f} \)
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\[
\text{support function of} \quad \text{“polar” of (T - v)}
\]

A lattice-free convex set \( T \) around fractional point \( v \)

From lattice-free convex set to \( \bar{f} \): [Johnson (1974)], [D., Wolsey (2010)], [Basu, Cornuéjols, Zambelli (2011)], [Conforti et al. (2015)]
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- **\( \bar{f} \)**: Subadditive and sublinear function
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\[
\begin{align*}
\text{Monoidal strengthening (Trivial lifting) } & \quad \uparrow \\
\text{and general lifting } & \quad (\text{Not necessarily unique}) \\
\bar{f} \quad \text{Subadditive and sublinear function} & \quad \uparrow \\
\text{support function of } & \quad \text{“polar” of } (T - v) \\
\text{A lattice-free convex set } T \text{ around fractional point } v & \quad \\
\end{align*}
\]

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One relationship via “intersection cuts” viewpoint of the lattice-free convex cuts for the set, \( \{ x \in \mathbb{Z}^m, z \in \mathbb{Z}^{n_1}_+, y \in \mathbb{R}^{n_2}_+, |x = b + Az + Gy\} \). Cuts in \((y, z)\)-space (Sketch):

<table>
<thead>
<tr>
<th>Subadditive function ((f))</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

A more concrete example of equivalence

- \( P := \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \) and \( S := P \cap \{ x \mid x_j \in \mathbb{Z} \ \forall i \in I \} \).

**Theorem ([Cornuéjols, Li (2002)])**

Let:

- **Split disjunctive closure**: \( \bigcap_{\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}} P^{\pi, \pi_0} = \text{intersection of all split cuts for all possible split disjunctions} \).
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- ▶ **Gomory mixed integer cut closure:** For any $\lambda \in \mathbb{R}^m$, generate GMI cut for $\{x \in \mathbb{R}_+^n \mid \lambda^T Ax = \lambda^T b, x_j \in \mathbb{Z} \ \forall i \in I\}$ and take the intersection of all these inequalities.
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Then:

Split disjunctive closure = Gomory mixed integer cut closure.
Section 4

Algebraic ideas
Reformulation-Linearization Technique

[Sherali Adams (1990)]
(Closely related to Lift-and-project) [Balas, Ceria, Cornuéjols (1993)]

Consider the binary:

\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \forall i \in [m] \\
\]
\[
x_j \in \{0, 1\} \quad \forall j \in [n_1]
\]
Reformulation-Linearization Technique

[Sherali Adams (1990)]
(Closely related to Lift-and-project) [Balas, Ceria, Cornuéjols (1993)]

Let's re-write binary MILPs as:

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i \in [m] \]

\[ x_j^2 = x_j \quad \forall j \in [n_1] \]
Reformulation-Linearization Technique

[Sherali Adams (1990)]
(Closely related to Lift-and-project) [Balas, Ceria, Cornuéjols (1993)]

For convenience let's write as:

\[ b_i - \sum_{j=1}^{n} a_{ij}x_j \geq 0 \ \forall i \in [m] \]

\[ x_j \geq 0 \ \forall j \in [n_1] \]

\[ 1 - x_j \geq 0 \ \forall j \in [n_1] \]

\[ x_j^2 = x_j \ \forall j \in [n_1] \]
(‘Standard’ RL Technique) Step 1: reformulation

Multiply linear constraints:

\[ b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0 \quad \forall i \in [m] \]

\[ x_j \geq 0 \quad \forall j \in [n_1] \]

\[ 1 - x_j \geq 0 \quad \forall j \in [n_1] \]

\[ x_j^2 = x_j \quad \forall j \in [n_1] \]
(‘Standard’ RL Technique) Step 1: reformulation

Multiply linear constraints:

\[
x_k \cdot \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \geq 0 \ \forall i \in [m], \forall k \in [n_1]
\]

\[
(1 - x_k) \cdot \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \geq 0 \ \forall i \in [m], \forall k \in [n_1]
\]

\[
x_k \cdot x_j \geq 0 \ \forall j \in [n_1], \forall k \in [n_1]
\]

\[
(1 - x_k) \cdot x_j \geq 0 \ \forall j \in [n_1], \forall k \in [n_1]
\]

\[
x_k \cdot (1 - x_j) \geq 0 \ \forall j \in [n_1], \forall k \in [n_1]
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x_j^2 = x_j \ \forall j \in [n_1]
\]
(‘Standard’ RL Technique) Step 1: linearization

Replace $x_j \cdot x_k$ by a new variables, say $w_{jk}$

\[
x_k \cdot \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \geq 0 \ \forall i \in [m], \forall k \in [n_1]
\]

\[
(1 - x_k) \cdot \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \geq 0 \ \forall i \in [m], \forall k \in [n_1]
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\[
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\]
(‘Standard’ RL Technique) Step 1: linearization

- Replace $x_j \cdot x_k$ by a new variable, say $w_{jk}$

\[
\begin{align*}
(b_i - \sum_{j=1}^{n} a_{ij} x_j) - \left( b_i x_k - \sum_{j=1}^{n} a_{ij} w_{jk} \right) & \geq 0 \ \forall i \in [m], \forall k \in [n_1] \\
(b_i - \sum_{j=1}^{n} a_{ij} x_j) - \left( b_i x_k - \sum_{j=1}^{n} a_{ij} w_{jk} \right) & \geq 0 \ \forall i \in [m], \forall k \in [n_1] \\
w_{jk} & \geq 0 \ \forall j \in [n_1], \forall k \in [n_1] \\
x_j - w_{jk} & \geq 0 \ \forall j \in [n_1], \forall k \in [n_1] \\
x_k - w_{jk} & \geq 0 \ \forall j \in [n_1], \forall k \in [n_1] \\
1 - x_k - x_j + w_{jk} & \geq 0 \ \forall j \in [n_1], \forall k \in [n_1] \\
w_{jj} & = x_j \ \forall j \in [n_1]
\end{align*}
\]

$\text{RLT1}(P)$
What's the point?

[Sherali Adams (1990)]

Let $P := \{x \in [0, 1]^{n_1} \times \mathbb{R}^{n_2} \mid Ax \leq b\}$.

Remember $P^{e_0,0} = \text{conv} \left( (P \cap \{x \mid x_j \leq 0\}) \cup (P \cap \{x \mid x_j \geq 1\}) \right)$.
Whats the point?  
[Sherali Adams (1990)]

Let \( P := \{ x \in [0, 1]^n \times \mathbb{R}^n | Ax \leq b \} \).

Remember \( P^{e_i,0} = \text{conv}\{ (P \cap \{ x | x_j \leq 0 \}) \cup (P \cap \{ x | x_j \geq 1 \}) \} \).

**Theorem ([Balas, Ceria, Cornuéjols (1993)])**

Let \( P, \text{RLT}_1(P), \) and \( P^{e_i,0} \) be as defined above. Then:

\[
\text{proj}_x(\text{RLT}_1(P)) = \bigcap_{j=1}^{n} P^{e_i,0}.
\]
Whats the point?
[Sherali Adams (1990)]

- Let $P := \{x \in [0, 1]^m \times \mathbb{R}^n | Ax \leq b\}$.
- Remember $P_{e^i,0} = \text{conv}\{(P \cap \{x | x_j \leq 0\}) \cup (P \cap \{x | x_j \geq 1\})\}$.

**Theorem ([Balas, Ceria, Cornuéjols (1993)])**

Let $P$, $RLT1(P)$, and $P_{e^i,0}$ be as defined above. Then:

$$\text{proj}_x(RLT1(P)) = \bigcap_{j=1}^{P_{e^i,0}}.$$

- The power of RLT comes from the multiplication of inequalities!
What's the point?
[Sherali Adams (1990)]

- Let \( P := \{ x \in [0, 1]^n_1 \times \mathbb{R}^n_2 \mid Ax \leq b \} \).
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- The process of multiplying and linearization applied only to \( x_j \geq 0 \) and \( 1 - x_j \geq 0 \), then we obtain the McCormick inequalities.
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Let $P$, $\text{RLT}_1(P)$, and $P^{e_i,0}$ be as defined above. Then:

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- The process of *multiplying and linearization applied only to* $x_j \geq 0$ *and* $1 - x_j \geq 0$, then we obtain the McCormick inequalities.
- This technique generalizes to polynomial optimization.
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- The process of *multiplying and linearization applied only to* \( x_j \geq 0 \) *and* \( 1 - x_j \geq 0 \), then we obtain the **McCormick inequalities**.
- This technique generalizes to **polynomial optimization**.
- This process can be strengthened by adding **implied semi-definite constraints**.
Semidefinite programming relaxation + RLT

\[
\left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) - \left( b_i x_k - \sum_{j=1}^{n} a_{ij} w_{ij} \right) \geq 0 \quad \forall i \in [m], \forall k \in [n_1]
\]

\[
\begin{array}{cccccc}
1 & x_1 & x_2 & \ldots & x_n \\
x_1 & w_{11} & w_{12} & \ldots & w_{1n} \\
x_2 & w_{21} & w_{22} & \ldots & w_{2n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
x_n & w_{n1} & w_{n2} & \ldots & w_{nn}
\end{array}
\]

\[
\begin{array}{cccccc}
\text{w}_{jk} \\
x_j - w_{jk} \\
x_k - w_{jk} \\
1 - x_k - x_j + w_{jk} \\
\text{w}_{jj}
\end{array} \geq \text{w}_{jj} = x_j \quad \forall j \in [n_1]
\]

\[
\begin{array}{cccccc}
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\]
Section 5

Relaxation based cuts
The main idea

- We would like to generate cuts valid for $P \cap \mathbb{Z}^n$, which is challenging in general.
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▶ We would like to generate cuts valid for $P \cap \mathbb{Z}^n$, which is challenging in general.

▶ We consider a relaxation of $P$, say $Q$ that is we find valid inequalities for $Q \cap \mathbb{Z}^n$, where $Q \supseteq P$. 

\[ Q \cap \mathbb{Z}^n, \]

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\[ P \quad \text{Q} \]
The main idea

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Some classic examples

- **Knapsack polytope.**

\[
\{ x \in \{0, 1\}^n \mid \sum_{j=1}^{n} a_j x_j \leq b \}.
\]

Some classic examples

- **Knapsack polytope.**

- **Mixing set.**

\[ \{(x, y) \in \{0, 1\}^n \times \mathbb{R}_+ \mid x_i + y \geq b_i \ \forall i \in [n] \} \].

Some classic examples

- **Knapsack polytope.**
- **Mixing set.**

流覆盖: \[ \left\{ (x, y) \in \{0, 1\}^n \times \mathbb{R}^n_+ \middle| \sum_{i=1}^n y_i \leq b, \ y_i \leq a_i x_i \ \forall i \in [n] \right\} . \]
Some classic examples

- **Knapsack polytope.**
- **Mixing set.**
- **Fixed charge network flow.**
- **Clique.** [Johnson, Padberg (1982)], [Atamtürk, Nemhauser, Savelsberg (2000)]

\[
\{ x \in \{0, 1\}^n \mid x_i + x_j \leq 1 \ \forall i, j \in [n] \times [n], \ i \neq j \}. 
\]
Some classic examples

- Knapsack polytope.
- Mixing set.
- Fixed charge network flow.
- Clique.

\[
\left\{ (x, w) \in \{0, 1\}^n \times \{0, 1\}^{\frac{(n)(n-1)}{2}} \mid w_{ij} = x_i x_j \quad \forall i, j \in [n] \times [n], \quad i \neq j \right\}.
\]

Connection to cuts for QCQPs. [Burer, Letchford (2009)]
Section 6

Measuring strength of cuts
Measuring strength of cuts - I

- **Does it produce a finite algorithm?**
  
  Pure integer: [Gomory (1958)], [Conforti, De Santis, Di Summa, Rinaldi (2021)]
  Mixed integer: [Dash et al. (2013)]
  Matching: [Chandrasekaran, Végh, Vempala (2016)]
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▶ Does it produce the convex hull?
  Matching polytope using Chvátal-Gomory: [Edmonds (1965)]
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- Approximation to the convex hull?
  Huge literature in CS theory.
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- Are they facet-defining for the relaxation?
  Group relaxation: [Gomory, Johnson (1972ab)], [Johnson (1974)],
  [Basu, Conforti, Cornuéjols, Zambelli (2010)], [Cornuéjols and Molinaro (2024)],
  [Basu, R. Hildebrand, Köppe (2014abcd)]
  [Basu, Hildebrand, Köppe, Molinaro (2013)],
  [Köppe, Zhou (2017)], [Di Summa (2020)]
Measuring strength of cuts - II

*Rank of a cut-plane procedure:*

- **Closure of cutting plane:** Add all cuts that can be generated by the cutting-plane procedure.
Measuring strength of cuts - II

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- **Closure may not be the convex hull.**

Theorem (Pure integer program)

Let $P$ be an arbitrary rational polyhedron. Then for Chvátal-Gomory cuts, we have the following:

- The rank is finite. [Schrijver (1980)]
- If $P \subseteq [0, 1]^n$, then the rank is bounded by $O(n^2 \log n)$. [Eisenbrand, Schulz (2003)]
- There exists a binary knapsack polytope whose rank is at least $\Omega(n^2)$. [Rothvoß, Sanitá (2017)]

Theorem

Let $P \subseteq [0, 1]^n$ be an arbitrary rational polyhedron. Then the rank of the RLT procedure is at most $n$. 

Huge literature on ranks of cutting-planes.
Measuring strength of cuts - II

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- If \( r \) is the smallest integer such that the \( r \)th closure is the convex hull, we say the rank is \( r \).
Measuring strength of cuts - II

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**Theorem**

Let \( P \subseteq [0, 1]^n \) be an arbitrary rational polyhedron. Then the rank of the RLT procedure is at most \( n \).
How do solvers select cuts to use?

- Maximize depth of cut: $\alpha^\top x^* - \beta \|\alpha\|_2$
- Cuts separating multiple known fractional point/point in relative interior or even interior.
- Parallelism between cuts/objective function.
- Sparsity.

Facet-defining or not?
How do solvers select cuts to use?

I do not know.
How do solvers select cuts to use?

But, here is a list of things that might matter:

- **Maximize depth of cut:**  \[
\frac{\alpha^T x^* - \beta}{\|\alpha\|_2}
\]

- Consider a point \( x^* \) that can be separated by the inequality: \( \alpha^T x \leq \beta \), for a packing problem.

- Suppose \( \alpha_1 > 0 \) and \( x_1^* = 0 \).

- Then setting \( \alpha_1 = 0 \) is a valid inequality (packing problem) and improves the depth of cut: However this cut is dominated by the original inequality!
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- **Maximize depth of cut**: \( \frac{\alpha^T x^* - \beta}{\|\alpha\|_2} \)


- Consider a point \( x^* \) that can be separated by the inequality: \( \alpha^T x \leq \beta \), for a *packing problem* [Shah, D. , Molinaro (2024)]

- Suppose \( \alpha_1 > 0 \) and \( x_1^* = 0 \).

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- **Cuts separating multiple known fractional point/point in relative interior or even interior**. [Fischetti, Salvagnin (2009)], [Turner, Berthold, Besançon, Koch (2023)]
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- **Sparsity**.
- **Facet-defining or not?**
  Closely related to *normalization for cut-generating LP*. [Conforti, Wolsey (2019)]
How many cuts to add?

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How many cuts to add?

- [Balas, Ceria, Cornuéjols, Natraj (1996)]
- [Shah, D., Molinaro (2024)]

- For instances enlight_hard and seymour1, where a moderate gap was closed (G between 20% and 50%) the tree size had a generally decreasing trend with some intermittent rises.

- In most instances, the total gap closed was small (G at most 20%). These instances did not have a decreasing trend. Instances from mas, gen-ip and neos, istanbul-no-cutoff and ran14x18-disj-8 often had large spikes and drops with every round of cut. The largest jump in tree size was seen in the case of supportcase26 where, for all of the 3 seeds, the tree size increased by more than 20 times within a single round. On the other hand, rmatr100-p10 and glass-sc showed no change in tree size in spite of closing some gap.

- Finally, in the case of fastxgemm-n2r6s0t2, pk1, markshare_4_0 and mad, the gap closed is 0 even after 10 rounds of cuts, but the tree sizes changed significantly in both directions. It must be noted that these instances clearly have high dual degeneracy which also contributes to variability in size.

An inference that can be drawn is that if the gap closed is small, change in tree size is difficult to predict, and often increases, possibly due to non-monotonicity. However, when a large enough gap is closed, a significant decrease in tree size may be expected. This is seen clearly in Fig. 6 where there are no data points with an increase in tree size when the gap closed exceeds 20%. Note that a very similar pattern is also seen in Fig. 4c for the randomly generated MKP instances.

Figure 6: Change in tree size and the gap closed across all instances, seeds and limits on rounds of cuts.
Some review papers

Thank You!