Introduction to cutting planes for mixed integer linear (nonlinear) programs

Santanu S. Dey

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Section 1

Introduction

Cuts: obtaining better dual bounds

Mixed integer linear program

$$\begin{aligned} z^{OPT} := \quad \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b && \text{(convex constraints)} \\ & x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}. && \text{(non-convex constraints)} \end{aligned}$$

Cuts: obtaining better dual bounds

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4. Improving LP dual bound by adding cutting-planes.

$$\begin{array}{rlll} z^{\text{LP+CUTS}} := & \max & c^\top x & \\ & \text{s.t.} & Ax \le b & \text{(convex constraints)} \\ & & \tilde{A}x \le \tilde{b} & \text{(valid for convex hull – Cuts)} \end{array}$$

Cuts: obtaining better dual bounds

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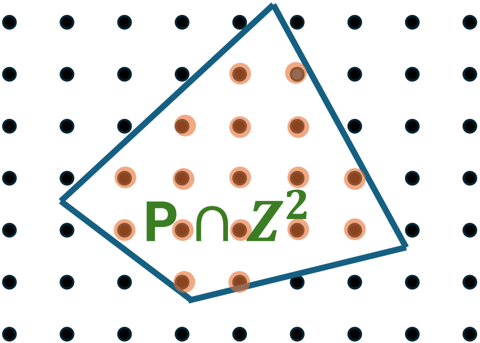
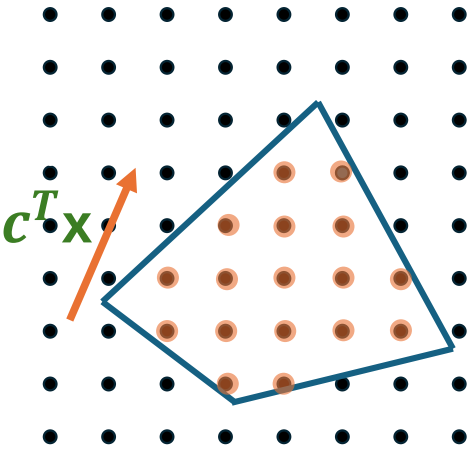
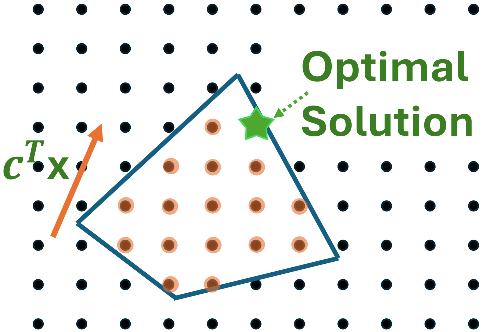
An integer program: feasible region



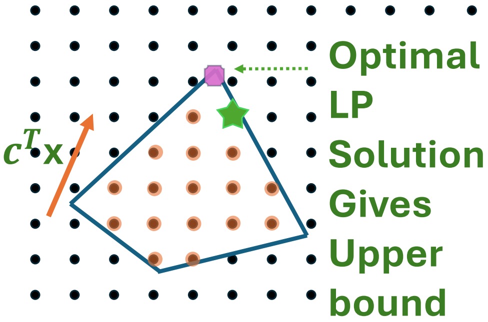
An integer program: objective function



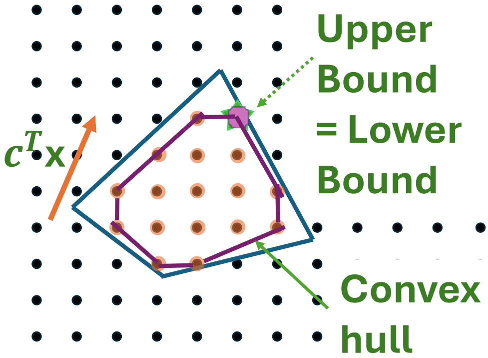
An integer program: optimal solution



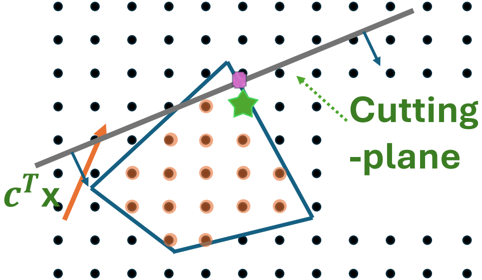
An integer program: dual bound from LP relaxation



An integer program: perfect dual bound from convex hull



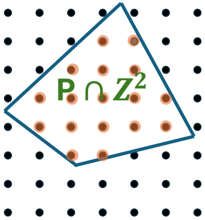
An integer program: improved dual bound using cutting-plane(s)

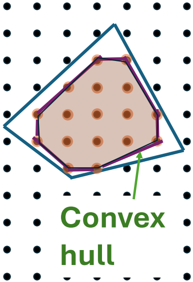


Why linear inequalities is a reasonable choice: Fundamental theorem of integer programming

Theorem ([Meyer (1974)])

Let $S := \{x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \mid Ax \leq b\}$. *If* A *and* b *is rational, then* $\operatorname{conv}(S)$ *is a rational polyhedron.*





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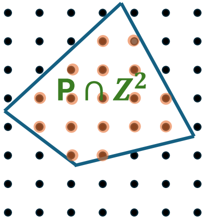
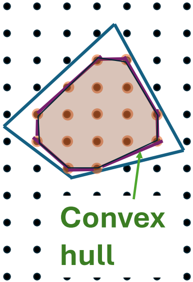




- Also adding linear cutting-plane, means we need to only solve modified LPs with dual simplex.
- Generalization of the above result for integer points in general convex set: [D., Morán (2013)]

How to generate cutting-planes?

- Geometric ideas: Split Disjunctive cuts, Chvátal-Gomory Cuts, maximal lattice-free cuts.
- Subadditive inequalities: Gomory mixed integer cut.
- Cuts using algebraic properties: Extended formulations.
- Cut from structured relaxations: Boolean quadric polytope, Clique cuts, Mixed integer rounding inequalities, Lifted cover, Flow cover, Mixing inequalities, ...
- Lifting: A technique to generate, rotate and strengthen inequalities. (Not covering this technique here)
- . . .

Section 2

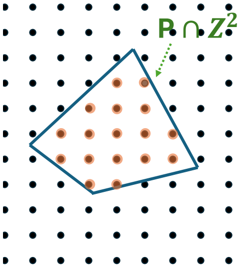
Geometric Ideas

2.1
Split disjunctive cuts

Split disjunctive cuts

[Balas (1979)][Cook, Kannan, Schrijver (1990)]

- Let $P \subseteq \mathbb{R}^n$ be a set and we are interested in obtaining valid inequality for $P \cap \mathbb{Z}^n$.



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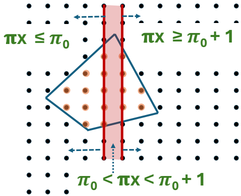
- Since

$$\mathbb{Z}^n \cap \underbrace{\{x \in \mathbb{R}^n \,|\, \pi_0 < \pi^\top x < \pi_0 + 1\}}_{\text{Split disjunctive set}} = \emptyset.$$



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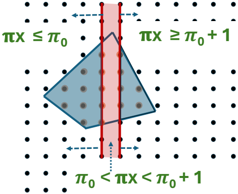

- If $\alpha^\top x \leq \beta$ is valid for:
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and therefore also for: $P \cap \mathbb{Z}^n$.

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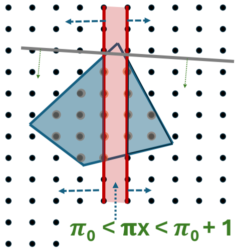

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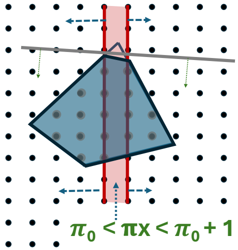

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Special-case: Chvátal-Gomory Cuts

[Gomory (1958)]

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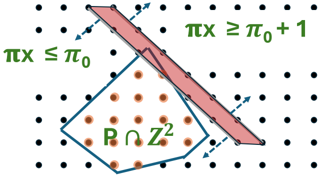

Follow-up work: [Schrijver (1980)], [Dadush, D., Vielma (2014)], [Cornuéjols, Lee (2018)]

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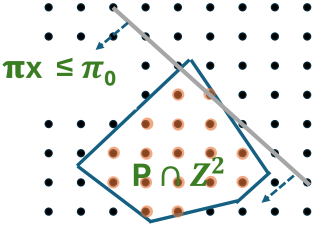

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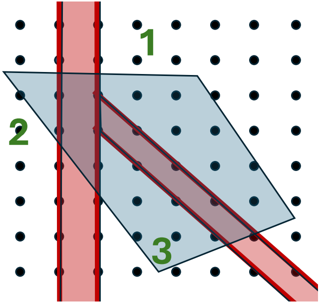
- *How is the valid inequality found?*
  - Valid inequality for $\text{conv}(P \setminus \text{int}(T))$.
  - Closed-form "formula"?

1.2
Generalizations of split disjunctive cuts

Types of lattice-free T sets I: non-convex

- *Asymmetric* [Dash, D., Günlük (2012)].



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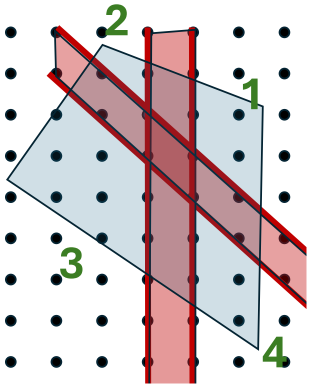

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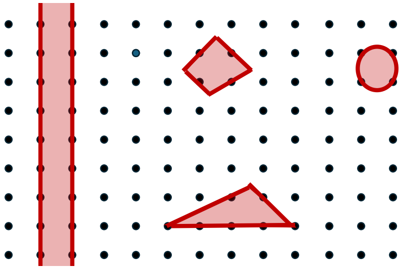

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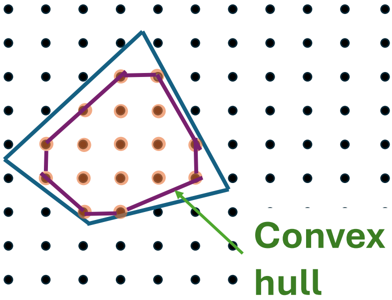

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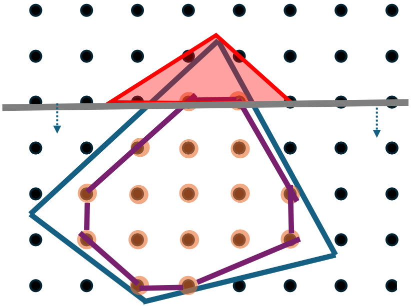

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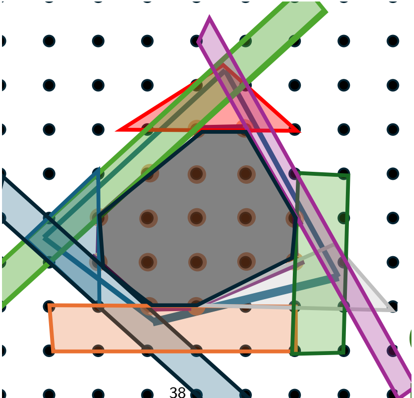
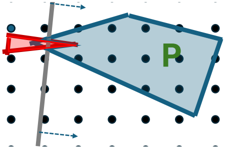


Maximal lattice-free convex set

- Clearly larger the lattice-free convex set T, we expect to find better inequality.



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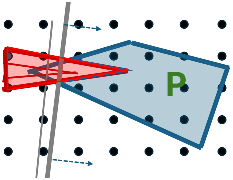

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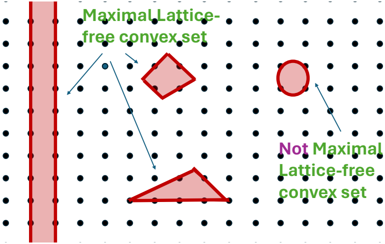
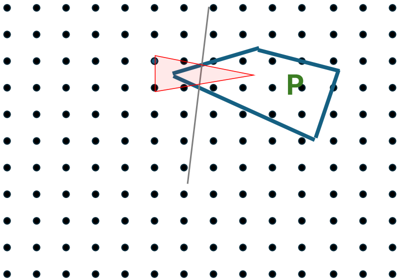
Theorem ([Lovász (1989)], [Basu, Conforti, Cornuéjols, Conforti (2010)])

All maximal lattice-free convex sets are polyhedral. Moreover, a full-dimension lattice-free convex set is maximal iff it is a lattice-free polyhedron with integer point in the relative interior of it facets.

Maximal lattice-free convex set



Generalization of maximal lattice-free sets



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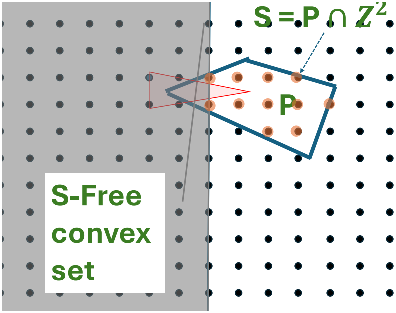

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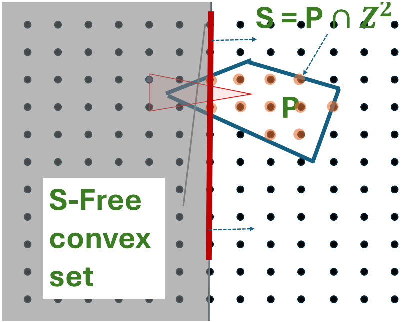

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Theorem ([D., Morán (2011)])

All maximal S-free convex sets are polyhedral.

Polyhedrality of maximal lattice-free sets is useful

- Let maximal lattice-free (or S-free) set be $T := \{x \in \mathbb{R}^n \,|\, (g^i)^\top x \geq h^i \;\; i \in [m]\}$.

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$$\bigvee_{i=1}^{m} P \cap \left\{ x \in \mathbb{R}^n \,|\, \underbrace{(g^i)^\top x \leq h^i}_{\text{complement of a facet of } T} \right\},$$

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$$\begin{aligned} \max_{\alpha,\beta} \quad & \alpha^\top x^* - \beta \\ \text{s.t.} \quad & \alpha x \leq \beta \text{ is valid for } \left(P \cap \{x \in \mathbb{R}^n \,|\, (g^i)^\top x \leq h^i\}\right) \;\; \forall i \in [m] \end{aligned}$$

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$$\begin{array}{llll} \max_{\alpha,\beta,\lambda,\mu} & \alpha^\top x^* - \beta & & \\ & \alpha^\top & = & (\lambda^i)^\top A + \mu^i \cdot (g^i)^\top \;\; \forall i \in [m] \\ \text{s.t.} & \beta & \geq & (\lambda^i)^\top b + \mu^i \cdot h^i \;\; \forall i \in [m] \\ & & & \lambda^i \geq 0, \mu^i \geq 0 \;\; \forall i \in [m] \end{array} \left.\vphantom{\begin{array}{l} a \\ a \\ a \end{array}}\right\} \text{Cone}$$

Normalization constraint: either bound α or β.

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- See [Balas, Perregaard: (2003)] for a method to generate cuts for split disjunctions with just one copy of variables (instead of two copies).

Final comments

- A major topic of study 2005-2015: [Andersen, Louveaux, Weismantel, Wolsey (2007)], [Borozan Cornuéjols (2009)], [D. Wolsey (2010)] [Del Pia Weismantel (2012)], ...

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- This is very general paradigm: See, for example,
  - Disjunctive ideas to get convex hull of QCQPs: [Tawarmalani, Richard, Chung (2010)], [D., Santana (2020)]
  - Intersection cuts for non-convex quadratically constrained quadratic programs. [Bienstock, Chen, Muñoz (2020)], [Muñoz, Serrano (2022)], [Chmiela, Muñoz, Serrano (2022)], [Muñoz, Paat, Serrano (2023)].

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- The real challenge is how to select the lattice-free set.

Section 3

Subadditive cutting-planes

A simple observation

- Subbaditive function: A function $f : \mathbb{R}^m \to \mathbb{R}$ is called subadditive if:

$$f(u) + f(v) \geq f(u + v) \text{ for all } u, v \in \mathbb{R}^m.$$

- Non-decreasing function: A function $f : \mathbb{R}^m \to \mathbb{R}$ is called non-decreasing if:

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Theorem ([Gomory, Johnson (1972ab)], [Jeroslow (1978)][Jeroslow (1979)], [Blair, Jeroslow (1982)])

$$\text{Let } S := \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} A^j x_j \geq b,\; x \in \mathbb{Z}^n \right\},$$

where $A^j \in \mathbb{R}^m$ for $j \in [n]$ and $b \in \mathbb{R}^m$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a subadditive function, non-decreasing, such that $f(0) = 0$, then

$$\sum_{j=1}^{n} f(A^j) x_j \geq f(b),$$

is a valid inequality for S.

Example of subadditive function

Consider the following set:

$$S := \left\{ x \in \mathbb{Z}^3_+ \;\middle|\; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 \geq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$:

$$f(u) = \lceil 0.5 \cdot (u_1 + u_2 + u_3) \rceil$$

This function is

- subadditive,
- non-decreasing,
- and $f(0) = 0$.

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So we have the following valid inequality for S:

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) x_1 + f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) x_2 + f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) x_3 \geq f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

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Equivalently:

$$x_1 + x_2 + x_3 \geq 2,$$

which is a facet-defining inequity for $\mathrm{conv}(S)$.

Mixed integer version

Theorem ([Gomory, Johnson (1972ab)])

Consider the set:

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where $A^j \in \mathbb{R}^m$ for $j \in [n]$ and $b \in \mathbb{R}^m$.

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  then

$$\sum_{j \in I} f(A^j) x_j + \sum_{j \in [n] \setminus I} \bar{f}(A^j) x_j \geq f(b),$$

is a valid inequality for S.

Mixed integer version - variants

Theorem ([Gomory, Johnson (1972)])

Consider the set:

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where $A^j \in \mathbb{R}^m$ for $j \in [n]$ and $b \in \mathbb{R}^m$. Let

- *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a sub-additive function, ~~non-decreasing~~, such that $f(0) = 0$, and*
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A very very special sub-additive function: Gomory mixed integer cut (GMIC)

[Gomory, Johnson (1972ab)]

- $S := \left\{(x, y) \in \mathbb{Z}^{n_1}_{+} \times \mathbb{R}^{n_2}_{+} \mid \sum_{j=1}^{n_1} a_j x_j + \sum_{i=1}^{n_2} d_i y_i = b\right\}.$

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- Let $\text{frc}(a) = a - \lfloor a \rfloor$.
- $f^{GMIC}(u) = \min\left\{\frac{\text{frc}(u)}{\text{frc}(b)}, \frac{1-\text{frc}(u)}{1-\text{frc}(b)}\right\}$, $\overline{f^{GMIC}}(u) = \begin{cases} u/\text{frc}(b) & u \geq 0 \\ (-u)/(1-\text{frc}(b)) & u \leq 0 \end{cases}$

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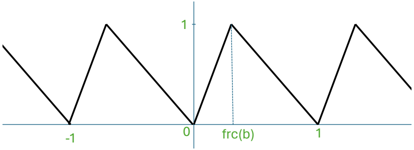
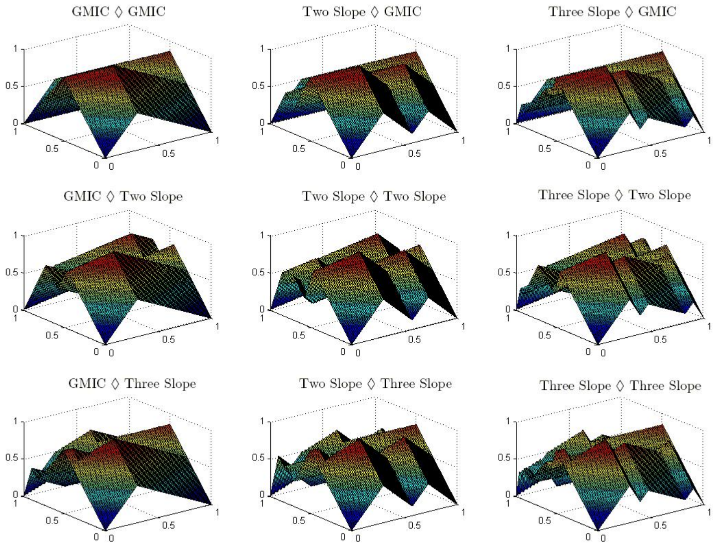


- Gomory-mixed integer cut:

$$\sum_{j \in [n_1], \text{frc}(a_j) \leq \text{frc}(b)} \frac{\text{frc}(a_j)}{\text{frc}(b)} x_j + \sum_{j \in [n_1], \text{frc}(a_j) \geq \text{frc}(b)} \frac{1-\text{frc}(a_j)}{1-\text{frc}(b)} x_j$$

$$\sum_{i \in [n_2], d_i \geq 0} \frac{d_i}{\text{frc}(b)} + \sum_{i \in [n_2], d_i \leq 0} \frac{-d_i}{1-\text{frc}(b)} \geq 1.$$

A zoo of subadditive functions



A zoo of subadditive functions

- Functions, functions, and more functions: [Letchford and Lodi (2002)], [Gomory, Johnson (2003)], [Dash, Günlük (2006)], [D., Richard (2008)], [Kianfar, Fathi (2009)], [Richard, Li, Miller (2009)], [D., Richard (2010)], [D., Richard, Li, Miller (2010)], [Chen (2011)], [Basu, Conforti, Paat (2018)], [Basu, Conforti, Di Summa (2020)] ...

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How good are these "subadditive cuts"?

Theorem ([Jeroslow (1978)], [Jeroslow (1979)], [Johnson (1973)], [Johnson (1974)], [Johnson (1979)])

Consider the set:

$$S := \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} A^j x_j \geq b,\; x_j \in \mathbb{Z}\; j \in I \right\},$$

where all the data is rational. Then the convex hull of S can be obtained using inequalities generated by non-decreasing, subadditive functions (with $f(0) = 0$).

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Only a particular type of subadditive functions called as Chvátal functions are necessary for the above result: [Blair, Jeroslow (1982)], [Basu, Martin, Ryan, Wang (2019)]

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where all the data is rational. Then the convex hull of S can be obtained using inequalities generated by non-decreasing, subadditive functions (with $f(0) = 0$).

Theorem (Wolsey [1981])

Consider the set:

$$S(b) := \left\{ x \in \mathbb{Z}^n_+ \;\middle|\; \sum_{j=1}^{n} A^j x_j = b, \right\}.$$

For A fixed, there is a finite list of subadditive functions that give the convex hull of $S(b)$ for all b.

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where all the data is rational. Then the convex hull of S can be obtained using inequalities generated by non-decreasing, subadditive functions (with $f(0) = 0$).

Theorem ([D., Morán, Vielma (2012)])

Consider the set:

$$S := \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} A^j x_j \succeq_K b,\;\; x_j \in \mathbb{Z}\; j \in I \right\},$$

where K is a proper cone and there exists a strictly feasible solution $\hat{x}$. Then the convex hull of S can be obtained using inequalities generated by non-decreasing (appropriately defined wrt K), subadditive functions (with $f(0) = 0$).

Follow-up: [Kocuk, Morán (2019)]

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One relationship via "intersection cuts" viewpoint of the lattice-free convex cuts for the set, $\{x \in \mathbb{Z}^m, z \in \mathbb{Z}_+^{n_1}, y \in \mathbb{R}_+^{n_2}, \,|\, x = b + Az + Gy\}$. Cuts in (y, z)-space (Sketch):

Subbadditive function (f)

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$$\downarrow \quad \left(T = \{x | \bar{f}(x - v) \leq 1\}\right)^{a}$$

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$\boxed{\text{Subbadditive function (f)}}$

$\downarrow$ $\left(\text{Slope of f: } \lim_{\epsilon \to 0^+} \frac{f(u\epsilon)}{\epsilon}\right)$

$\boxed{\bar{f}}$ Subadditive and sublinear function

$\downarrow$ $(T = \{x | \bar{f}(x - v) \leq 1\})$ [a]

$\boxed{\text{A lattice-free convex set } T \text{ around fractional point } v}$

From $\bar{f}$ to lattice-free convex set: [Borozan Cornuéjols (2009)], [Conforti et al.(2015)]

[a] With proper scaling of $\bar{f}$

Any connection between maximal lattice-free convex cuts and subadditive cuts?

- We can obtain the convex hull using maximal lattice-free convex cuts and also subadditive cuts — is there a connection? YES!

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$\uparrow$ Monoidal strengthening (Trivial lifting) and general lifting (Not necessarily unique)

$\boxed{\bar{f}}$ Subadditive and sublinear function

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A lattice-free convex set T around fractional point v

From $\bar{f}$ to f: Monoidal Strengthening [Balas, Jeroslow (1980)], [D., Wolsey (2010)], Uniqueness: [Basu, Cornuéjols, Koéppe (2012)], [Campelo et al. (2013)], [Basu, Averkov (2014)], [Basu, Paat (2015)], [Basu, D., Paat (2019)]

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A more concrete example of equivalence

- $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and $S := P \cap \{x \mid x_j \in \mathbb{Z} \; \forall i \in I\}$.

Theorem ([Cornuéjols, Li (2002)])

Let:

- *Split disjunctive closure:* $\bigcap_{\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}} P^{\pi,\pi_0}$ *= intersection of all split cuts for all possible split disjunctions .*

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Then:

Split disjunctive closure $=$ Gomory mixed integer cut closure.

Section 4

Algebraic ideas

Reformulation-Linearization Technique

[Sherali Adams (1990)]
(Closely related to Lift-and-project) [Balas, Ceria, Cornuéjols (1993)]

Consider the binary:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \ \forall i \in [m]$$
$$x_j \in \{0,1\} \ \forall j \in [n_1]$$

Reformulation-Linearization Technique

[Sherali Adams (1990)]
(Closely related to Lift-and-project) [Balas, Ceria, Cornuéjols (1993)]

Lets re-write binary MILPs as:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \ \forall i \in [m]$$

$$x_j^2 = x_j \ \forall j \in [n_1]$$

Reformulation-Linearization Technique

[Sherali Adams (1990)]
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For convenience lets write as:

$$
\begin{aligned}
b_i - \sum_{j=1}^{n} a_{ij}x_j \geq 0 \ \forall i \in [m] \\
x_j \geq 0 \ \forall j \in [n_1] \\
1 - x_j \geq 0 \ \forall j \in [n_1] \\
x_j^2 = x_j \ \forall j \in [n_1]
\end{aligned}
$$

('Standard' RL Technique) Step 1: reformulation

Multiply linear constraints:

$$\begin{aligned} b_i - \sum_{j=1}^{n} a_{ij} x_j &\geq 0 \ \forall i \in [m] \\ x_j &\geq 0 \ \forall j \in [n_1] \\ 1 - x_j &\geq 0 \ \forall j \in [n_1] \\ x_j^2 &= x_j \ \forall j \in [n_1] \end{aligned}$$

('Standard' RL Technique) Step 1: reformulation

Multiply linear constraints:

$$
\begin{aligned}
x_k \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) &\geq 0 \; \forall i \in [m], \forall k \in [n_1] \\
(1 - x_k) \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) &\geq 0 \; \forall i \in [m], \forall k \in [n_1] \\
x_k \cdot x_j &\geq 0 \; \forall j \in [n_1], \forall k \in [n_1] \\
(1 - x_k) \cdot x_j &\geq 0 \; \forall j \in [n_1], \forall k \in [n_1] \\
x_k \cdot (1 - x_j) &\geq 0 \; \forall j \in [n_1], \forall k \in [n_1] \\
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- *Replace $x_j \cdot x_k$ by a new variables, say w_{jk}*

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\end{array}\right\} \mathrm{RLT1}(\mathrm{P}$$

Whats the point?

[Sherali Adams (1990)]

- Let $P := \{x \in [0,\ 1]^{n_1} \times \mathbb{R}^{n_2} \mid Ax \leq b\}$.
- Remember $P^{e^j,0} = \operatorname{conv}\{(P \cap \{x \mid x_j \leq 0\}) \cup (P \cap \{x \mid x_j \geq 1\})\}$.

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Theorem ([Balas, Ceria, Cornuéjols (1993)])

Let P, RLT1(P), and $P^{e_j,0}$ be as defined above. Then:

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- The process of *multiplying and linearization applied only to* $x_j \geq 0$ *and* $1 - x_j \geq 0$, then we obtain the McCormick inequalities.

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- This technique generalizes to polynomial optimization.
- This process can be strengthened by adding implied semi-definite constraints.

Semidefinite programming relaxation + RLT

$$
\begin{array}{rcl}
\left(b_i x_k - \sum_{j=1}^n a_{ij} w_{ij}\right) & \geq & 0 \ \forall i \in [m], \forall k \in [n_1] \\
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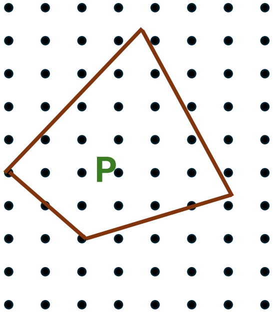
\begin{bmatrix}
1 & x_1 & x_2 & \dots & x_n \\
x_1 & w_{11} & w_{12} & \dots & w_{1n} \\
x_2 & w_{21} & w_{22} & \dots & w_{2n} \\
\dots & \dots & \dots & \dots & \dots \\
x_n & w_{n1} & w_{n2} & \dots & w_{nn}
\end{bmatrix} & \succeq & 0.
\end{array}
$$

Section 5

Relaxation based cuts

The main idea

- We would like generate cuts valid for $P \cap \mathbb{Z}^n$, which is challenging in general.



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- We would like generate cuts valid for $P \cap \mathbb{Z}^n$, which is challenging in general.
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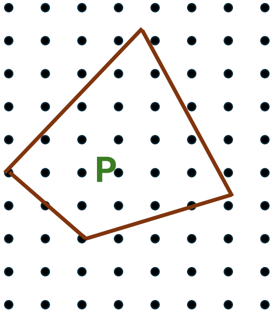

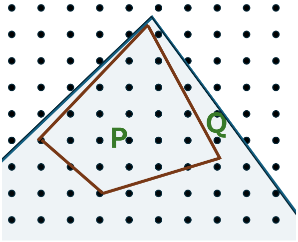

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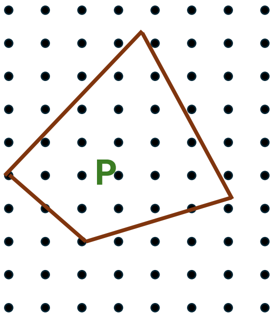

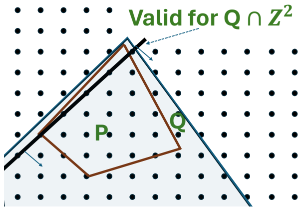

Some classic examples

- Knapsack polytope.

$$\left\{ x \in \{0,1\}^n \;\middle|\; \sum_{j=1}^{n} a_j x_j \le b \right\}.$$

Cover inequalities and other inequalities [Wolsey (1975)], [Balas (1975)], [Hammer, Johnson,Peled (1975)], Weismantel (1997), lifted cover inequalities [Zemel (1978)], [Balas, Zemel (1984)], [Crowder, Johnson, Padberg (1983)], Mixed binary: [Van Roy, Wolsey (1986)], [Gu, Nemhauser, Savelsberg (2000)], [Richard, de Farias Jr, Nemhauser (2003ab)] General Integer and continuous variables Knapsack constraint: [Atamtürk (2003)],[Atamtürk (2004)]

Some classic examples

- Knapsack polytope.
- Mixing set.

$$\{(x,y)\in\{0,1\}^n\times\mathbb{R}_+ \mid x_i+y\geq b_i\ \forall i\in[n]\}\,.$$

[Günlük, Pochet (2001)] Special case when $n=1$: Mixed integer rounding (MIR) inequalities.($\equiv$ Gomory mixed integer cut in closure.) [Nemhauser, Wolsey (1990)], [Dash, Günlük, Lodi (2010)], Extensions: [Marchand, Wolsey (1999)], [Van Vyve (2005)], [Atamtürk, Günlük (2010)], [D., Wolsey (2010)], Chance-constrained programming: [Luedtke, Ahmed, Nemhauser (2010)], [Küçükyavuz 92012)], [Kılınç-Karzan, Küçükyavuz, Lee (2022)]

Some classic examples

- Knapsack polytope.
- Mixing set.
- Fixed charge network flow. Submodularity: [Wolsey (1989)], [Atamtürk, S. Küçükyavuz, and B. Tezel (2017)], Flow cover: [Padberg, Van Roy, Wolsey (1985)], [Gu, Nemhauser, Savelsberg (2000)], Network design: [Atamtürk, Günlük (2007)]

$$\text{Flow cover: } \left\{ (x, y) \in \{0,1\}^n \times \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} y_i \leq b,\;\; y_i \leq a_i x_i \;\forall i \in [n] \right\}.$$

Some classic examples

- Knapsack polytope.
- Mixing set.
- Fixed charge network flow.
- Clique. [Johnson, Padberg (1982)], [Atamtürk, Nemhauser, Savelsberg (2000)]

$$\left\{x \in \{0,1\}^n \,|\, x_i + x_j \leq 1 \;\forall i,j \in [n] \times [n],\;\; i \neq j\right\}.$$

Some classic examples

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- Clique.
- Boolean quadric polytope. [Padberg (1989)], [Boros, Hammer (1993)], [De Simone (1996)] Cut polytope: [Barahona, Mahjoub (1986)], [Sherali, Lee, Adams (1995)] Review: [Letchford (2022)]

$$\left\{(x, w) \in \{0,1\}^n \times \{0,1\}^{\frac{(n)(n-1)}{2}} \,\middle|\, w_{ij} = x_i x_j \ \forall i, j \in [n] \times [n],\ i \neq j\right\}.$$

Connection to cuts for QCQPs.[Burer, Letchford (2009)]

Section 6

Measuring strength of cuts

Measuring strength of cuts - I

- Does it produce a finite algorithm?
  Pure integer: [Gomory (1958)], [Conforti, De Santis, Di Summa, Rinaldi (2021)] Mixed integer: [Dash et al. (2013)], Matching: [Chandrasekaran, Végh, Vempala (2016)]

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- Approximation to the convex hull?
  Huge literature in CS theory.
- Are they facet-defining for the relaxation?
  Group relaxation: [Gomory, Johnson (1972ab)], [Johnson (1974)], [Gomory, Johnson (2003)], [D., Richard, Miller (2010)], [Basu, Hildebrand, Molinaro (2018)], [Basu, Conforti, Cornuéjols, Zambelli (2010)], [Cornuéjols and Molinaro (2024)], [Basu, R. Hildebrand, Köppe (2014abcd)] [Basu, Hildebrand, Köppe, Molinaro (2013)], [Köppe, Zhou (2017)], [Di Summa (2020)]

Measuring strength of cuts - II

Rank of a cut-plane procedure:

- Closure of cutting plane: Add all cuts that can be generated by the cutting-plane procedure.

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- Closure of cutting plane: Add all cuts that can be generated by the cutting-plane procedure.
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Theorem (Pure integer program)

Let P be an arbitrary rational polyhedron. Then for Chvátal-Gomory cuts, we have the following:

- *The rank is finite. [Schrijver (1980)]*

Measuring strength of cuts - II

Rank of a cut-plane procedure:

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- So we may obtain the closure of the closure, this is the second closure.
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Theorem

Let $P \subseteq [0,\ 1]^n$ be an arbitrary rational polyhedron. Then the rank of the RLT procedure is at most n.

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I do not know.

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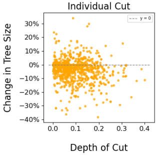
But, here is a list of things that might matter:

- *Maximize depth of cut*: $\frac{\alpha^\top x^* - \beta}{\|\alpha\|_2}$
  Not always the best [Andreello, Caprara, Fischetti (2007)], [Amaldi, Coniglio, Gualandi (2014)].
  - Consider a point x^* that can be separated by the inequality: $\alpha^\top x \leq \beta$, for a *packing problem*.
  - Suppose $\alpha_1 > 0$ and $x_1^* = 0$.
  - Then setting $\alpha_1 = 0$ is a valid inequality (*packing problem*) and improves the depth of cut: However this cut is dominated by the original inequality!

[Shah, D. , Molinaro (2024)]



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  Variants of depth of cut: [Wesselmann, Suhl (2007)], Volume: [Basu, Conforti, Di Summa, Zambelli (2019)], [Zhou (2023)]

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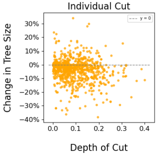

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- *Sparsity*. [Amaldi, Coniglio, Gualandi (2014)], [D., Molinaro, Wang (2015)], [D., Molinaro, Wang (2018)]

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- *Sparsity*.
- *Facet-defining or not?*
  Closely related to *normalization for cut-generating LP*. [Conforti, Wolsey (2019)]

How many cuts to add?

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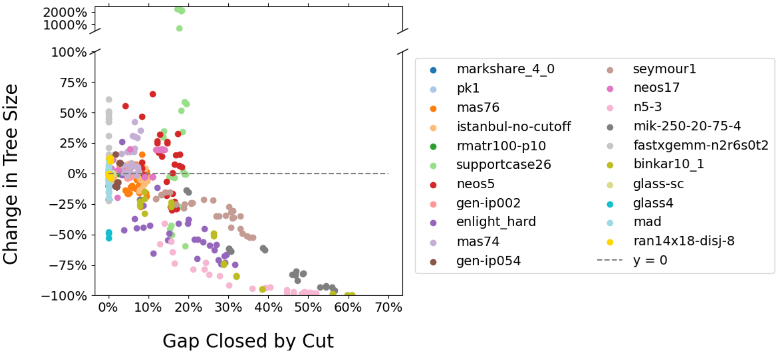

Some review papers

- Theoretical challenges towards cutting-plane selection. D., Molinaro (2018).
- Light on the infinite group relaxation. Basu, Hildebrand, Koëppe (2016).
- Lifting techniques for mixed integer programming, Richard (2011).
- The group-theoretic approach in mixed integer programming. D., Richard (2010).
- Cutting planes in integer and mixed integer programming. Marchand, Martin, Weismantel, Wolsey (2002).
- Progress in linear programming-based algorithms for integer programming: an exposition. Johnson, Nemhauser, Savelsbergh (2000).

Thank You!