Convexification of Bilinear Terms over Network Polytopes

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Joint work with:

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Outline of the talk

Part I: Introduction

Part II: Aggregation Method

Part III: Convexification for $m = 1$

Part IV: Convexification for $m > 1$

Part V: Computational Experiments
Problem definition

Consider

\[ \mathcal{S} = \{(x; y; z) \in \Sigma \times \Delta_m \times \mathbb{R}^\kappa \mid x_i y_j = z_{ij}, \ \forall (i, j) \in N \times M\}, \]

where

- \( \Sigma = \{x \in \mathbb{R}^n \mid Ex \geq f, \ 0 \leq x \leq u\} \) is a network polytope described by the flow-balance and arc capacity constraints

- \( \Delta_m = \{y \in \mathbb{R}_+^m \mid 1^T y \leq 1\} \) is a simplex
  - Naturally imposed: SOS1
  - Reformulated: \( \forall \)-representation of polytopes
Applications

Structure of set $S$ appears in various optimization models in different application areas such as:

- Fixed-charge network flow problems
- Transportation problem with conflicts
- Bilevel network flow problems
- Network interdiction problems
- Optimization via decision diagrams
- ...
Applications

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- Transportation problem with conflicts
- Bilevel network flow problems
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- Optimization via decision diagrams
- ...

Studying convexification of $S$ can lead to cutting planes that can tighten existing relaxations and improve dual bounds.
Motivation

McCormick relaxation [McCormick, 1976]

► Replace \( z = xy \) with \( z \geq 0, z \geq x + y - 1, z \leq x, z \leq y \) for unit box domain on \( x \) and \( y \).

► Provides convex hull over box domains [Al-Khayal & Falk, 1983].

► Often leads to weak relaxations when the domain is more general [Luedtke et. al., 2012], [Gupte et. al., 2013].

Disjunctive programming [Balas, 1985] or special structure RLT [Sherali et. al., 1998]

► Describes convex hull in a higher dimension.

► Can be computationally expensive due to large size of extended formulation.

► Uses separation to generate cuts in the original space of variables.

► Does not provide explicit forms of facet-defining inequalities in the original space of variables.
Design a systematic procedure to obtain an explicit form of the convex hull description in the original space of variables.
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Part III: Convexification for $m = 1$

Part IV: Convexification for $m > 1$

Part V: Computational Experiments
An aggregation procedure

We use a specialized aggregation procedure, called *Extended Cancel & Relax (EC&R)*, to obtain valid inequalities for $\text{conv}(S)$

[Davarnia, Richard, Tawarmanali, 2017]
An aggregation procedure

We use a specialized aggregation procedure, called *Extended Cancel & Relax (EC&R)*, to obtain valid inequalities for $\text{conv}(S)$

[Davarnia, Richard, Tawarmanali, 2017]

**Step 1: Assign weights**

- Pick a bilinear constraint with weight $\pm 1$ (base constraint)

  $$ (\pm 1) \times (x_k y_l - z_{kl} = 0) $$

- Pick linear constraints from $\Sigma$ with the following weights:

  $$ \beta \left( \begin{array}{c} y_1 \\ \vdots \\ y_m \\ 1 - \sum_{j \in M} y_j \end{array} \right) \times \left( \begin{array}{c} E_t \mathbf{x} \geq f_t \\ x_i \geq 0 \\ 1 - x_i \geq 0 \end{array} \right) $$
An aggregation procedure

**Step 2:** Aggregate the above weighted inequalities such that at least $|\mathcal{I}|$ bilinear terms cancel.
An aggregation procedure

**Step 2**: Aggregate the above weighted inequalities such that at least $|\mathcal{I}|$ bilinear terms cancel

**Step 3**: Relax the remaining bilinear term using McCormick bounds or bilinear constraints

- Replace $x_i y_j$ with
  - $u_i y_j$
  - $x_i$
  - $z_{ij}$

- Replace $-x_i y_j$ with
  - $0$
  - $x_i + u_i y_j - u_i$
  - $-z_{ij}$
Theorem

A linear description of $\text{conv}(S)$ is given by:

- the inequalities defining $\Sigma$,
- the inequalities defining $\Delta_m$,
- all EC&R inequalities.

Proof sketch:

- Observe that $S$ is bounded.
- The vertices of $S$ are such that $y = e_i$ or $y = 0$.
- The restriction of $S$ to $y = e_i$ ($y = 0$) is polyhedral.
- The convex hull of $S$ can be obtained in higher dimension using disjunctive programming.
- The rays of the projection cone of the disjunctive programming formulation have structure.
- The components of the rays can be interpreted as "dual" weights on the initial constraints of the system that "cancel" product variables.
Theorem

A linear description of \( \text{conv} (S) \) is given by:

- the inequalities defining \( \Sigma \),
- the inequalities defining \( \Delta_m \),
- all EC&R inequalities.

Proof sketch:

- Observe that \( S \) is bounded.
- The vertices of \( S \) are such that \( y = e_i \) or \( y = 0 \).
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Level-1 generalization of McCormick

McCormick relaxation:

\[ xy = z \]

over

\[ x \in [0, u] \]

\[ y \in [0, 1] \]
Level-1 generalization of McCormick

McCormick relaxation:

\[ xy = z \]

over

\[ x \in [0, u] \]
\[ y \in [0, 1] \]

Level-1 generalization:

\[ x_i y = z_i, \quad \forall i \in A \]

over

\[ x \in \Sigma = \{ x \in \mathbb{R}^n | Ex \geq f, 0 \leq x \leq u \} \]
\[ y \in [0, 1] \]
Determining aggregation weights

**Proposition**

Let \( \mathbf{a}^\top \mathbf{x} + by + \mathbf{c}^\top \mathbf{z} \geq d \) be a non-trivial facet-defining inequality of \( \text{conv}(S) \) that is obtained from the EC&R procedure. Then, the weights \( \beta \) of all network constraints used in the aggregation are equal to 1.

Proof sketch:
- The projection problem for the disjunctive programming formulation has the following form:
  \[
  E^\top \pi = e_k.
  \]
- The coefficient matrix is TU.
- The RHS is a unit vector.
- The weights \( \pi \) are non-negative.
Determining aggregation weights

Proposition

Let $a^T x + by + c^T z \geq d$ be a non-trivial facet-defining inequality of $\text{conv}(S)$ that is obtained from the EC&R procedure. Then, the weights $\beta$ of all network constraints used in the aggregation are equal to 1.

Proof sketch:

- The projection problem for the disjunctive programming formulation has the following form

\[
\begin{bmatrix}
E^T & -E^T & \pm I & \pm I & I & -I
\end{bmatrix} \pi = \pm e^k.
\]

- The coefficient matrix is TU
- The RHS is a unit vector
- The weights $\pi$ are non-negative
Graphical structure of EC&R inequalities

**Theorem**

Consider

- $S$ with $m = 1$ defined over a network $G = (V, A)$,
- $a^\top x + b y + c^\top z \geq d$ to be a non-trivial facet-defining inequality of $\text{conv}(S)$ that is obtained from the EC&R procedure.
- $\mathcal{I}$ to be the set of flow-balance constraints used in the aggregation.

Then,

- The nodes corresponding to constraints in $\mathcal{I}$ form a tree in $G$. 

Proof sketch (part I):

- From previous proposition, to cancel a bilinear term $x_i y$ for some $i \in A$, we need to use 2 flow-balance constraints at nodes $t(i)$ and $h(i)$

\[
1 \times y \times \left( \sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)
\]

\[
1 \times y \times \left( \sum_{j \in \delta^+(h(i))} x_j - \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j - x_i \geq f_{h(i)} \right)
\]

- The nodes whose flow-balance constraints are used in the aggregation must be adjacent
Proof sketch (part II):

- This subnetwork must be connected.

- **Contradiction:** for a disconnected subnetwork, the basis has the following form.

\[
\begin{bmatrix}
\pm E_1 & \pm I_1 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
\pm E_2 & \pm I_2 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
\pm e^i \\
0 \\
\end{bmatrix}.
\]

- Columns in second part are linearly dependent.
Example

Consider set $S$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint 

$$-x_{1,5}y + z_{1,5} = 0.$$
Example

Consider set $S$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint $-x_{1,5}y + z_{1,5} = 0$.

$$1 \times (-x_{1,5}y + z_{1,5} = 0)$$
Consider set $S$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint 

$$-x_{1,5}y + z_{1,5} = 0.$$ 

1. $1 \times (-x_{1,5}y + z_{1,5} = 0)$
2. $y \times (-x_{4,1} - x_{2,1} + x_{1,5} \geq f_1)$
3. $y \times (x_{4,1} + x_{4,3} - x_{8,4} \geq f_4)$
4. $1 - y \times (-x_{8,4} \geq -f_8)$
5. $y \times (x_{3,7} - x_{4,3} - x_{2,3} \geq f_3)$
Example

Consider set $S$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint $-x_{1,5}y + z_{1,5} = 0$.

\begin{align*}
1 \times (-x_{1,5}y + z_{1,5} = 0) \\
y \times (-x_{4,1} - x_{2,1} + x_{1,5} \geq f_1) \\
y \times (x_{4,1} + x_{4,3} - x_{8,4} \geq f_4) \\
1 - y \times (-x_{8,4} \geq -f_8) \\
y \times (x_{3,7} - x_{4,3} - x_{2,3} \geq f_3)
\end{align*}

\[ -x_{4,1}y - x_{2,1}y + x_{3,7}y - x_{2,3}y - x_{8,4} - (f_1 + f_4 + f_8 + f_3)y + z_{1,5} + f_8 + 0x_{1,5}y + 0x_{8,4}y + 0x_{4,1}y + 0x_{4,3}y \geq 0 \]
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Level-2 generalization of McCormick

McCormick relaxation:

$$xy = z$$

over

- $x \in [0, u]$
- $y \in [0, 1]$
Level-2 generalization of McCormick

McCormick relaxation:

\[ xy = z \]

over

\[ x \in [0, u] \]
\[ y \in [0, 1] \]

Level-2 generalization:

\[ x_i y_j = z_{ij}, \quad \forall i \in A, j \in M \]

over

\[ x \in \Sigma = \{ x \in \mathbb{R}^n \mid Ex \geq f, \ 0 \leq x \leq u \} \]
\[ y \in \Delta_m = \{ y \in \mathbb{R}_+^m \mid 1^\top y \leq 1 \} \]
A more complicated structure

The coefficient matrix in the projection problem for the disjunctive programming formulation has the following form

\[
\begin{bmatrix}
E^T & 0 & \ldots & 0 & -E^T & \pm I & \pm I & 0 & \ldots & 0 \\
0 & E^T & \ldots & 0 & -E^T & \pm I & 0 & \pm I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & E^T & -E^T & \pm I & 0 & 0 & \ldots & \pm I \\
\end{bmatrix}
\]

The TU property does not hold.
The aggregation weights for facet-defining EC&R inequalities are not necessarily 1.
A more complicated structure

The coefficient matrix in the projection problem for the disjunctive programming formulation has the following form

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0 & E^T & \ldots & 0 & -E^T & \pm I & 0 & \pm I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & E^T & -E^T & \pm I & 0 & 0 & \ldots & \pm I \\
\end{bmatrix}
\]

- The TU property does not hold.
- The aggregation weights for facet-defining EC&R inequalities are not necessarily 1.
A class of facet-defining inequalities

Consider the class of non-trivial facet-defining EC&R inequalities of $\text{conv}(S)$ with pairwise cancellation property.

**Definition**

An EC&R inequality has **pairwise cancellation** property if each cancellation of bilinear terms is obtained by aggregation two constraints.
Proposition

Let $a^T x + by + c^T z \geq d$ be a non-trivial facet-defining inequality of $\text{conv}(S)$ that is obtained from the EC&R procedure through pairwise cancellation. Then, the weights $\beta$ of all network constraints used in the aggregation are equal to 1.
Determining aggregation weights

Proposition

Let $\mathbf{a}^\top \mathbf{x} + b y + \mathbf{c}^\top \mathbf{z} \geq d$ be a non-trivial facet-defining inequality of $\text{conv}(\mathcal{S})$ that is obtained from the EC&R procedure through pairwise cancellation. Then, the weights $\beta$ of all network constraints used in the aggregation are equal to 1.

Proof sketch:

- The projection problem for the disjunctive programming formulation has the following form

\[
\begin{bmatrix}
\pm 1 & 0 & \cdots & 0 \\
\{0, \pm 1\} & \pm 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\{0, \pm 1\} & \{0, \pm 1\} & \cdots & \pm 1
\end{bmatrix}
\]

\[
\pi = \pm \mathbf{e}^1
\]

- All (positive) weights must be 1.
New definitions for network structures

Definition
Consider set $S$ defined over network $G = (V, A)$. We define a parallel network $G^j = (V^j, A^j)$ for $j \in \{1, \ldots, m\}$ to be a replica of $G$ that represents the multiplication of network constraints with $y_j$ during the aggregation procedure.
New definitions for network structures

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider set $S$ defined over network $G = (V, A)$. We define a <strong>parallel network</strong> $G^j = (V^j, A^j)$ for $j \in {1, \ldots, m}$ to be a replica of $G$ that represents the multiplication of network constraints with $y_j$ during the aggregation procedure.</td>
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<td>Consider $S$ defined over network $G = (V, A)$. Let $\hat{G}^j = (\hat{V}^j, \hat{A}^j)$, for $j = 1, 2$, be a subnetwork of parallel network $G^j$.</td>
</tr>
</tbody>
</table>

- We say that subnetworks $\hat{G}^1$ and $\hat{G}^2$ are **vertically connected through connection node** $v \in V$ if the replica of $v$ in parallel network $G^j$ is adjacent to a node of subnetwork $\hat{G}^j$ for $j = 1, 2$. |

- We say that subnetworks $\hat{G}^1$ and $\hat{G}^2$ are **vertically connected through connection arc** $a \in A$ if the replica of $a$ in parallel network $G^j$ is incident to a node of subnetwork $\hat{G}^j$ for $j = 1, 2$. |
Graphical structure of EC&R inequalities

**Theorem**

Consider

- $S$ defined over a network $G = (V, A)$,
- $\mathbf{a}^\top \mathbf{x} + b \mathbf{y} + \mathbf{c}^\top \mathbf{z} \geq d$ to be an EC&R facet-defining inequality of $\text{conv}(S)$ with the pairwise cancellation property.
- $I^j$ to be the set of flow-balance constraints used in the aggregation multiplied with $y_j$.
- $J$ to be the set of flow-balance constraints used in the aggregation multiplied with $1 - \sum_{j=1}^{m} y_j$.
- $K$ to be the set of variable bound constraints used in the aggregation multiplied with $1 - \sum_{j=1}^{m} y_j$.

Then,

- The nodes corresponding to $I^j$ form a forest $F^j$ in $G^j$.
- The forests $F^j$ are vertically connected through the connection nodes in $J$ and connection arcs in $K$. 
Graphical structure of EC&R inequalities

Proof sketch (part I):

From previous proposition, to cancel a bilinear term $x_i y_j$ for some $i \in A$ and $j \in M$, one of the following cases should occur:

- Two flow-balance constraints at nodes $t(i)$ and $h(i)$ are used in the aggregation:

\[
1 \times y_j \times \left( \sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)
\]

\[
1 \times y_j \times \left( \sum_{j \in \delta^+(h(i))} x_j - \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j - x_i \geq f_{h(i)} \right)
\]

- The nodes whose flow-balance constraints are used in the aggregation must be adjacent, forming a forest structure in parallel network $G^j$. 
Graphical structure of EC&R inequalities

Proof sketch (part II):

- Two flow-balance constraints at nodes $t(i)$ and $h(i)$ are used in the aggregation:

$$
1 \times y_j \times \left( \sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)
$$

$$
1 \times \left( 1 - \sum_{k=1}^{m} y_k \right) \times \left( - \sum_{j \in \delta^+(h(i))} x_j + \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j + x_i \geq -f_{h(i)} \right)
$$

- One of the nodes whose flow-balance constraint is used in the aggregation must be a connection node.
Graphical structure of EC&R inequalities

Proof sketch (part III):

- A flow-balance constraint at node $t(i)$ and a bound constraint for variable $x_i$ are used in the aggregation:

$$1 \times y_j \times \left( \sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)$$

$$1 \times \left( 1 - \sum_{k=1}^{m} y_k \right) \times (x_i \geq 0)$$

- The arc whose bound constraint is used in the aggregation must be a connection arc.
Example

Consider set $S$ with $y_1 + y_2 \leq 1$ defined over the following network. Assume that we are interested in finding EC&R inequalities obtained through pairwise cancellation with base constraint $x_{1,5}y_1 - z_{1,5} = 0$. 
Example

Consider set $S$ with $y_1 + y_2 \leq 1$ defined over the following network. Assume that we are interested in finding EC&R inequalities obtained through pairwise cancellation with base constraint $x_{1,5}y_1 - z_{1,5} = 0$. 

![Diagram of network $G^1$](image1)

![Diagram of network $G^2$](image2)
Example

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Consider set $S$ with $y_1 + y_2 \leq 1$ defined over the following network. Assume that we are interested in finding EC&R inequalities obtained through pairwise cancellation with base constraint $x_{1,5}y_1 - z_{1,5} = 0$. 

---

**Diagram Description:**

- $G^1$: A network with nodes 1, 2, 3, 4, 5, 6, 8, and 7, connected by arrows indicating flow or transitions.
- $G^2$: A network similar to $G^1$ but with a different orientation or labeling of the nodes.

---

**Graphical Representation:**

- Node 1 is connected to nodes 2 and 5 in $G^1$.
- Node 2 is connected to nodes 3 and 4 in $G^1$.
- Node 3 is connected in a loop, with a dotted arrow from 3 to itself.
- Node 4 is connected to nodes 6 and 8 in $G^1$.
- Node 5 is connected to node 1 in $G^1$.

- Node 1 is connected to nodes 2 and 5 in $G^2$.
- Node 2 is connected to nodes 3 and 4 in $G^2$.
- Node 3 is connected to nodes 8 and 4 in $G^2$.
- Node 4 is connected to nodes 6 and 8 in $G^2$.
- Node 5 is connected to node 1 in $G^2$. 

---

**Network Details:**

- The network $G^1$ and $G^2$ are likely used to illustrate the concept of pairwise cancellation and the constraints associated with the base constraint $x_{1,5}y_1 - z_{1,5} = 0$. The diagrams help visualize the flow and possible paths or cycles within the network to derive inequalities.
Consider set $S$ with $y_1 + y_2 \leq 1$ defined over the following network. Assume that we are interested in finding EC&R inequalities obtained through pairwise cancellation with base constraint $x_{1,5}y_1 - z_{1,5} = 0$. 
Example

Aggregate the corresponding constraints with appropriate weights.

\[ 1 \times (x_{1,5}y_1 - z_{1,5} = 0) \]
Aggregate the corresponding constraints with appropriate weights.

\[
1 \times (x_{1,5}y_1 - z_{1,5} = 0)
\]

\[
y_1 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1)
\]

\[
y_1 \times (\neg x_{2,1} - x_{2,3} + x_{6,2} \geq -f_2)
\]

\[
y_1 \times (\neg x_{6,2} \geq -f_6)
\]

\[
y_1 \times (\neg x_{8,4} \geq -f_8)
\]
Aggregate the corresponding constraints with appropriate weights.

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1 \times (x_{1,5}y_1 - z_{1,5} = 0)
\]

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y_1 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1)
\]

\[
y_1 \times (-x_{2,1} - x_{2,3} + x_{6,2} \geq -f_2)
\]

\[
y_1 \times (-x_{6,2} \geq -f_6)
\]

\[
y_1 \times (-x_{8,4} \geq -f_8)
\]

\[
y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1)
\]

\[
y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4)
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Aggregate the corresponding constraints with appropriate weights.

\[ 1 \times (x_{1,5}y_1 - z_{1,5} = 0) \]
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\[ y_1 \times (-x_{6,2} \geq -f_6) \]
\[ y_1 \times (-x_{8,4} \geq -f_8) \]
\[ y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \]
\[ y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4) \]
\[ 1 - y_1 - y_2 \times (-x_{2,3} - x_{4,3} + x_{3,7} \geq -f_3) \]
Example

Aggregate the corresponding constraints with appropriate weights.

\[ 1 \times (x_{1,5}y_1 - z_{1,5} = 0) \]
\[ y_1 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \]
\[ y_1 \times (-x_{2,1} - x_{2,3} + x_{6,2} \geq -f_2) \]
\[ y_1 \times (-x_{6,2} \geq -f_6) \]
\[ y_1 \times (-x_{8,4} \geq -f_8) \]
\[ y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \]
\[ y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4) \]
\[ 1 - y_1 - y_2 \times (-x_{2,3} - x_{4,3} + x_{3,7} \geq -f_3) \]
\[ 1 - y_1 - y_2 \times (-x_{8,4} \geq -u_{8,4}) \]
Example

The aggregated bilinear inequality is

\[- y_1 x_{4,5} - y_1 x_{3,7} + y_1 x_{4,3}
\quad - y_2 x_{1,5} + y_2 x_{4,5} + y_2 x_{2,3} - y_2 x_{3,7}
\quad + (f_1 + f_2 + f_3 + f_6 + f_8 - u_{8,4}) y_1
\quad + (f_1 + f_3 + f_4 - u_{8,4}) y_2 - z_{1,5}
\quad + x_{3,7} - x_{2,3} - x_{4,3} - x_{8,4} - f_3 + u_{8,4}
\quad + 0 x_{1,5} y_1 + 0 x_{2,1} y_1 + 0 x_{6,2} y_1 + 0 x_{8,4} y_1
\quad + 0 x_{2,3} y_1 + 0 x_{4,1} y_2 + 0 x_{4,3} y_2 + 0 x_{8,4} y_2 \geq 0.\]

Relaxing the remaining bilinear terms will yield linear EC&R inequalities.
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Part IV: Convexification for $m > 1$

Part V: Computational Experiments
Application I: Fixed-charge network flow problems

\[
\begin{align*}
\min & \quad \sum_{i \in S} \sum_{j \in D} \left( \left( c_{ij} + \frac{t_{ij}}{\epsilon_{ij}} \right) x_{ij} + t_{ij} y_{ij} - \frac{t_{ij}}{\epsilon_{ij}} z_{ij} \right) \\
x_{ij} y_{ij} &= z_{ij}, \quad \forall i \in S, j \in D \\
\sum_{j \in D} x_{ij} &\leq s_i, \quad \forall i \in S \\
\sum_{i \in S} x_{ij} &\geq d_j, \quad \forall j \in D \\
0 &\leq x_{ij} \leq u_{ij}, \quad \forall i \in S, j \in D \\
\sum_{i \in S} \sum_{j \in D} y_{ij} &\leq b, \\
0 &\leq y_{ij} \leq 1, \quad \forall i \in S, j \in D
\end{align*}
\]
Application I: Fixed-charge network flow problems

- Used settings from [Rebennack, Nahapetyan, Pardalos, (2009)]

- Number of nodes in bipartite graph: \{50, 100\}

- Breakpoint value: \{0.2, 0.5\}

- 10 instances for each size category

- Number of \(y\) variables: 20% of the arc number

- Used Gurobi to solve different relaxations

- Used EC&R inequalities for up to 3 constraint aggregations

- Used separation to add most violated cuts
Application I: Fixed-charge network flow problems

<table>
<thead>
<tr>
<th>Node #</th>
<th>Frac.</th>
<th>Solver</th>
<th>Tree EC&amp;R</th>
<th>RLT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Gap</td>
<td>Time</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Opt. Time</td>
<td>Root Gap</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>467.42</td>
<td>0.62</td>
<td>0.75</td>
</tr>
<tr>
<td>50</td>
<td>0.5</td>
<td>186.7</td>
<td>0.63</td>
<td>0.8</td>
</tr>
<tr>
<td>100</td>
<td>0.2</td>
<td>≥ 1000</td>
<td>0.41</td>
<td>0.49</td>
</tr>
<tr>
<td>100</td>
<td>0.5</td>
<td>≥ 1000</td>
<td>0.57</td>
<td>0.66</td>
</tr>
</tbody>
</table>

The table shows average results over 10 instances for each size category (row)
Application II: Transportation problem with conflicts

\[
\begin{align*}
\min & \quad \sum_{i \in S} \sum_{j \in D} \left( c_{ij} x_{ij} + \sum_{k \in K} r_{ij}^k z_{ij}^k \right) \\
& \quad x_{ij} y_k = z_{ij}^k, \quad \forall i \in S, j \in D, k \in K \\
& \quad \sum_{j \in D} x_{ij} \leq s_i, \quad \forall i \in S \\
& \quad \sum_{i \in S} x_{ij} \geq d_j, \quad \forall j \in D \\
& \quad 0 \leq x_{ij} \leq u_{ij}, \quad \forall i \in S, j \in D \\
& \quad \sum_{k \in L} y_k \leq 1, \quad \forall L \in C \\
& \quad y_k \in \{0, 1\}, \quad \forall k \in K
\end{align*}
\]
Application II: Transportation problem with conflicts

- Used settings from [Vancroonenburg, Della Croce, Goossens, Spieksma, (2014)]

- Number of nodes in bipartite graph: \{50, 100\}

- Number of transportation services: \{20, 30\}

- 10 instances for each size category

- Number of pairwise conflicts: 10\% of total pairwise combinations

- Used Gurobi to solve different relaxations

- Used EC&R inequalities for up to 3 constraint aggregations

- Used separation to add most violated cuts
### Application II: Transportation problem with conflicts

<table>
<thead>
<tr>
<th>Node #</th>
<th>Service #</th>
<th>Solver</th>
<th>Forest EC&amp;R</th>
<th>RLT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Opt. Time</td>
<td>Root Gap</td>
<td>Gap</td>
</tr>
<tr>
<td>50</td>
<td>20</td>
<td>70.74</td>
<td>0.1</td>
<td>0.53</td>
</tr>
<tr>
<td>50</td>
<td>30</td>
<td>341.2</td>
<td>0.19</td>
<td>0.54</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>1981.57</td>
<td>0.1</td>
<td>0.44</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>≥ 5000</td>
<td>0.18</td>
<td>0.49</td>
</tr>
</tbody>
</table>

- The table shows average results over 10 instances for each size category (row)
Conclusion

Summary:

▶ Developed a convexification method for bilinear set $S$ to obtain facet-defining inequalities in the original space of variables.
▶ Showed that for $m = 1$, these inequalities correspond to tree structures in the underlying graph.
▶ Showed that for case with $m > 1$, these inequalities correspond to special forest structures in the underlying graph.
▶ Presented computational results to show the effectiveness of the developed methods in improving dual bounds.

Reference: