The Treewidth-Convex Hull Theorem and DP for Cut Generation in MINLP

Sourabh K Choudhary 1  Santanu S. Dey 1  Nikolaos V. Sahinidis 1
1Georgia Institute of Technology

Summary
We provide an alternative proof for tree-width convex hull theorem. Next, in a column generation setting, we use dynamic programming to solve pricing problem for solving cut generating problems for non-linear problems with multilinear intermediates.

Unconstrained Binary Problem

\[ \min_{b \in \{0,1\}^n} \sum_{T \subset N} c_T \prod_{i \in T} b_i, \]

Unconstrained binary optimization has many applications ranging from finance, communication, theoretical physics to economics and machine learning.

Graph and Tree-Decomposition

![Graph and Tree-Decomposition](image)

Figure 1. Tree-decomposition of a graph

Unconstrained Binary Problem

- Theorem: Consider an UBP with treewidth \(d\) and number of variables \(n\). This problem can be reframed as an LP with \(O(n^2d)\) nonnegative variables and constraints. [2].
- Contribution: We show that the LP can be converted into a DP formulation and hence the integrality can be proven. A corollary is that the LP is Totally Dual Integral (TDI).

Multilinear Intermediates and Adding Cuts

- Next, consider a multilinear intermediate
  \[ z = L(x) = \sum_{T \subset N} c_T \prod_{i \in T} x_i \]
  where \( N = \{1, 2, 3, \ldots, n\}\) which is defined over a box \( H = \prod_i [l_i, u_i] \).
  - Given a relaxation solution \((x^*, z^*)\), the dual to cut generation LP is:
    \[ \min_{\lambda} \sum_{i \in T} \lambda_i L(v^i) \]
    \[ \sum_{i \in T} \lambda_i v^i = x^* \]
    \[ \sum_{i \in T} \lambda_i = 1 \]
    \[ \lambda_i \geq 0 \]
  - Let \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \) be the dual optimal to the first and second constraints. We add the cut \( z \geq ax + b \) if \( z^* < ax^* + b \).

The Framework

- We propose a column generation strategy to solve the dual to cut generation LP.
- Proposed DP strategy for pricing: Solve the unconstrained binary problem \( L(b) = \pi^T b - \pi_0 \) using dynamic programming.

Computational Results

<table>
<thead>
<tr>
<th>Type</th>
<th>Time(s)</th>
<th>Better by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>95%-55%</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>95%-55%</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>95%-55%</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>95%-55%</td>
</tr>
</tbody>
</table>

Example instances

We proposed a new proof for treewidth convex hull theorem and a framework for solving CGLP in NLPs. Future work involves integrating this within the Branch and Bound (B&B) tree of standard solvers.

Conclusion and Future Direction

References