The Treewidth-Convex Hull Theorem and DP for Cut Generation in MINLP

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Summary

We provide an alternative proof for tree-width convex hull theorem. Next, in a column generation setting, we use dynamic programming to solve pricing problem for solving cut generating problems for non-linear problems with multilinear intermediates.

Unconstrained Binary Problem

$$\min_{\mathbf{b}\in\{0,1\}^n} \sum_{T\subseteq N} c_T \prod_{i\in T} b_i,$$

Unconstrained binary optimization has many applications ranging from finance, communication, theoretical physics to economics and machine learning.

Graph and Tree-Decomposition

$$\min b_1b_2b_3 - 4b_2b_3b_4b_5 + 2b_2b_5 + 4b_3b_6$$

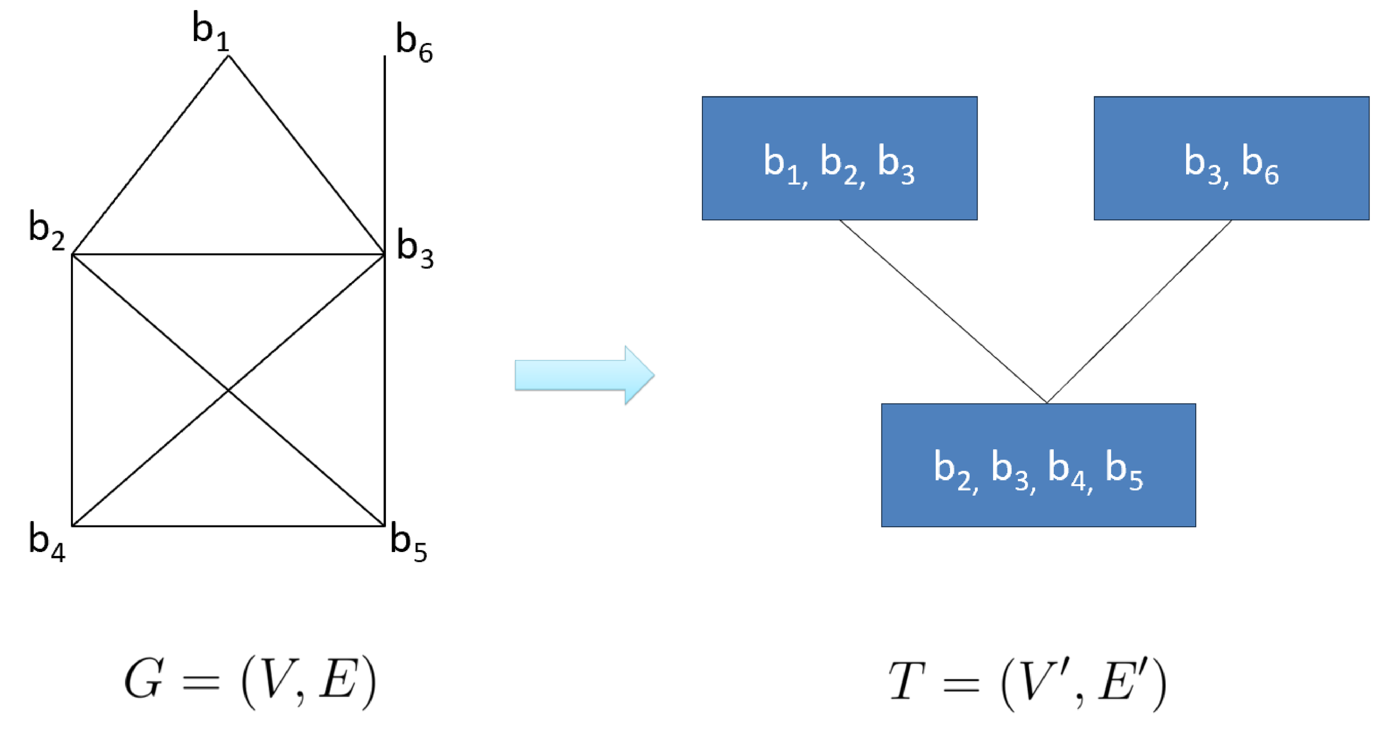

Figure 1. Tree-decomposition of a graph

Resulting Simplification

- **Theorem: Consider an UBP with treewidth d and number of variables n. This problem can be reframed as an LP with $O(n2^d)$ nonnegative variables and constraints.** [2].
- **Contribution:** We show that the LP can be converted into a DP formulation and hence the integrality can be proven. **A corollary is that the LP is Totally Dual Integral (TDI).**

Multilinear Intermediates and Adding Cuts

- Next, consider a multilinear intermediate
$$z = L(x) = \sum_{T\subseteq N} c_T \prod_{i\in T} x_i$$
where $N = \{1, 2, 3, ..., n\}$ which is defined over a box $\mathcal{H} = \prod_{j=1}^n [l_j, u_j]$.
- Given a relaxation solution (x^*, z^*), the dual to cut generation LP is: [1]:
$$\min_{\lambda} \sum_{i\in I} \lambda_i L(v^i)$$
$$\sum_{i\in I} \lambda_i v^i = x^*$$
$$\sum_{i\in I} \lambda_i = 1$$
$$\lambda_i \geq 0$$
- Let $a \in R^n$ and $b \in R$ be the dual optimal to the first and second constraints. We add the cut $z \geq ax + b$ if $z^* < ax^* + b$.

The Framework

- We propose a column generation strategy to solve the dual to cut generation LP.
- Proposed DP strategy for pricing: Solve the unconstrained binary problem $L(\mathbf{b}) - \pi^T\mathbf{b} - \pi_0$ using dynamic programming.

Computational Results

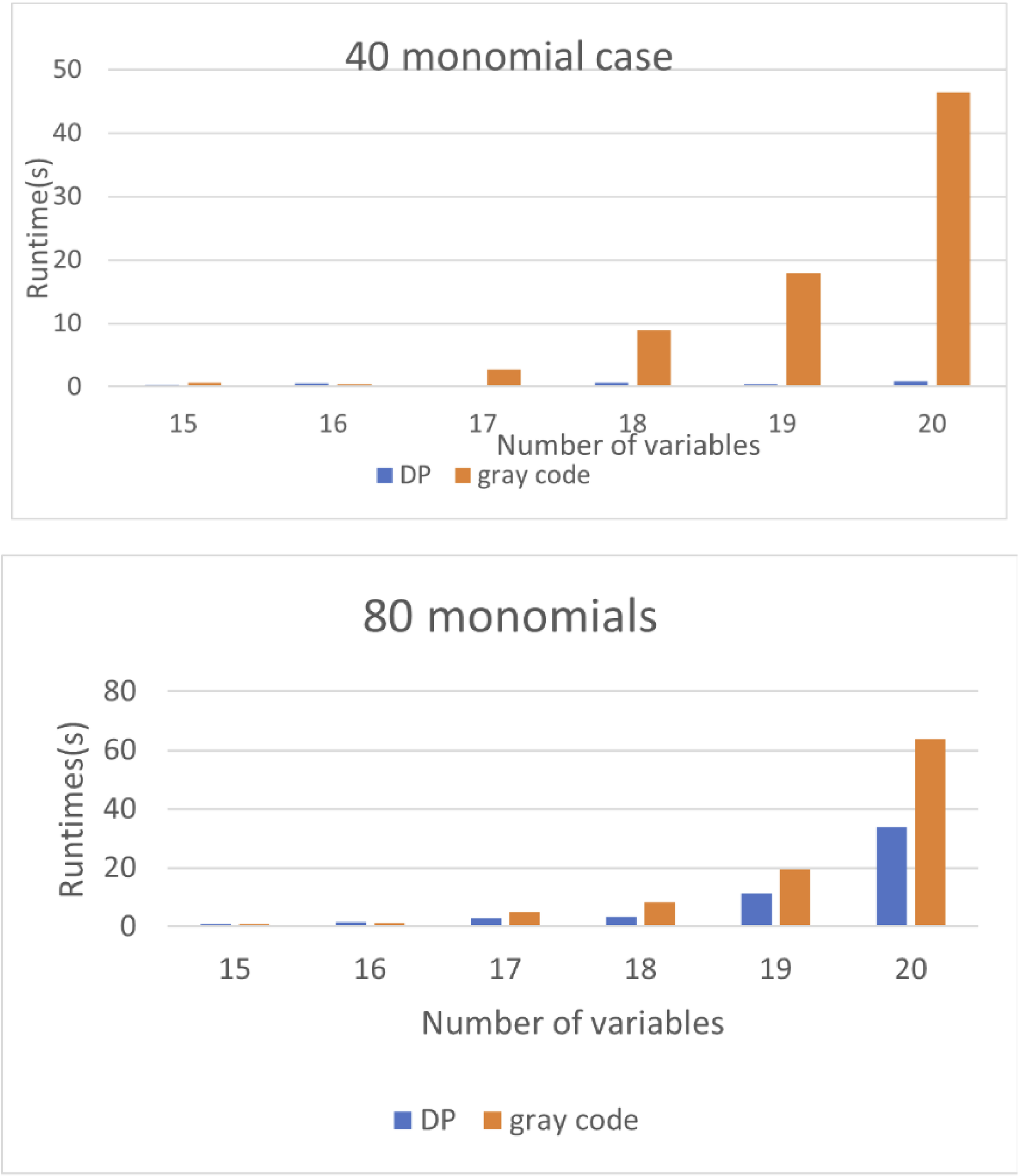

Figure 2. Comparison of time taken to solve CGLP

Example instances

n	d	Basic relaxation with our cuts	Time(s) to add cuts	BARON LB (20 min)	Better by
Type1[a]					
25	10	-14804	95	-42949	65.5%
26	10	-15622	111	-40100	61.0%
27	10	-16852	140	-47023	64.2%
28	10	-17862	149	-45524	60.8%
29	10	-18560	169	-56249	67.0%
30	10	-20286	184	-58543	65.3%
Type2[b]					
25	10	-12682	76	-23246	45.4%
26	10	-13481	93	-24864	45.8%
27	10	-14178	111	-26130	45.7%
28	10	-14973	124	-26358	43.2%
29	10	-15733	163	-30518	48.4%
30	10	-16506	163	-33234	50.3%

[a] Unconstrained NLP.
[b] Linearly constrained NLP.
* Instances generated from a known tree-decomposition.

Conclusion and Future Direction

We proposed a new proof for treewidth convex hull theorem and a framework for solving CGLP in NLPs. Future work involves integrating this within the Branch and Bound (B&B) tree of standard solvers.

References

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