New Sequence-Independent Lifting Techniques for Cutting Planes and When They Induce Facets

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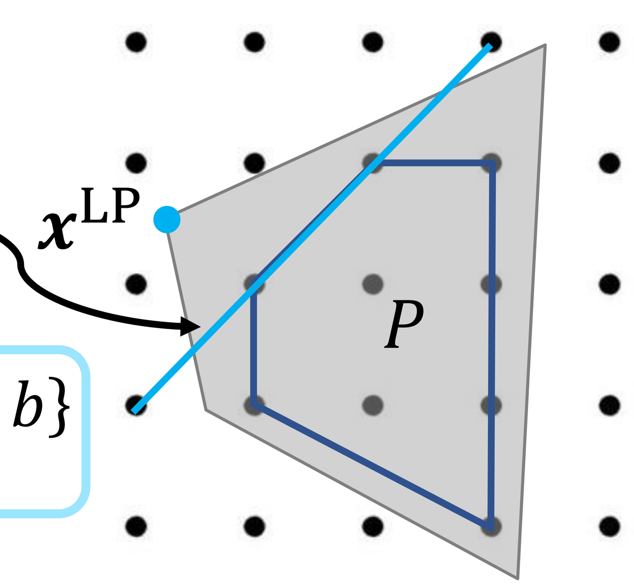
Highlights

- *Generalization of Gu-Nemhauser-Savelsbergh lifting technique (and correction of their generalization that yields invalid cuts).*
- *Characterization of when our new lifting function yields facet-defining cuts.*
- *Experiments with new cover cut families.*

Knapsack Polytopes and Cover Cuts

Facet-defining cut
Gold-standard for cuts



$P = \text{conv}\{x \in \{0,1\}^n : \sum_i a_i x_i \le b\}$
Knapsack polytope

$16x_1 + 14x_2 + 13x_3 + 9x_4 + \cdots + a_n x_n \le 44$

Items 1 – 4 are too heavy → enforce $x_1 + x_2 + x_3 + x_4 \le 3$

$C \subseteq \{1, \dots, n\}$ is a cover if $\sum_{j \in C} a_j > b$
C is a minimal cover if $\sum_{j \in C \setminus i} a_j \le b \ \forall i \in C$
Minimal cover cut: $\sum_{j \in C} x_j \le |C| - 1$

Sequence-Independent Lifting

Lifted cover cut: $\sum_{j \in C} x_j + \sum_{j \notin C} \pi_j x_j \le |C| - 1$.

Wolsey (1977): If $g \le f$ is *superadditive*,

$$\sum_{j \in C} x_j + \sum_{j \notin C} g(a_j) x_j \le |C| - 1$$

is a valid cut; g is **sequence-independent.**

Gu-Nemhauser-Savelsbergh (GNS) (2000) explicitly constructed a sequence-independent lifting function g; very easy to compute (see figure).

μ_h = weight of h heaviest items in C.
λ = excess weight of C.
ρ_h = excess weight of C if heaviest item replaced with a copy of $(h+1)$st heaviest item.

Piecewise-Constant (PC) Lifting

Theorem: *If $\mu_1 - \lambda \ge \rho_1$, one can use any slope (uniformly) to get a superadditive lifting fn.*

Generalizes GNS lifting and corrects an error in GNS's proposed generalization which yields invalid cuts.

PC Lifting: slope-zero piecewise-constant lifting function; all other slopes dominated by PC + GNS.

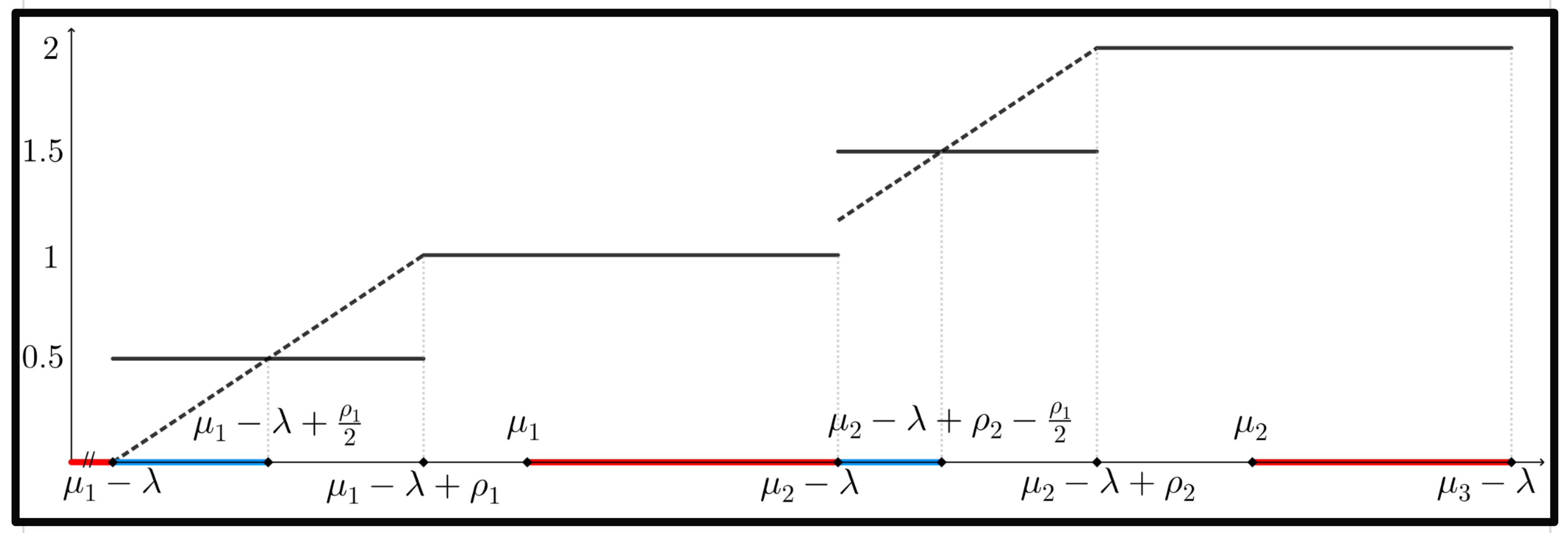


Figure. PC lifting (solid lines) vs GNS lifting (replace with dashed sloped segments).

Facet-Defining Cuts from PC lifting

Theorem: *If $\{a_j : j \notin C\}$ is contained in the blue and red intervals with at least 3 coeffs. in the leftmost blue interval, PC lifting dominates GNS lifting and is facet-defining:*

$\sum_{j \in C} x_j + \sum_{j \notin C} \text{PC}(a_j) x_j \le |C| - 1$ ***is a facet of P.***

- Proof uses Balas-Zemel (1978) characterization of facets of P that arise from lifting (BZ does not give a tractable way of deriving such cuts).
- *First set of conditions for facet-defining sequence-independent liftings that are efficiently computable from the underlying cover.*

$16x_1 + 14x_2 + 13x_3 + 9x_4 + 9x_5 + 10x_6 + 11x_7 + 23x_8 \le 44$

GNS: $x_1 + x_2 + x_3 + x_4 + \frac{1}{6}x_5 + \frac{1}{3}x_6 + \frac{1}{2}x_7 + \frac{4}{3}x_8 \le 3$

PC: $x_1 + x_2 + x_3 + x_4 + \frac{1}{2}(x_5 + x_6 + x_7) + \frac{3}{2}x_8 \le 3$

Here, PC strictly dominates GNS and is facet-defining.

Experimental Evaluation

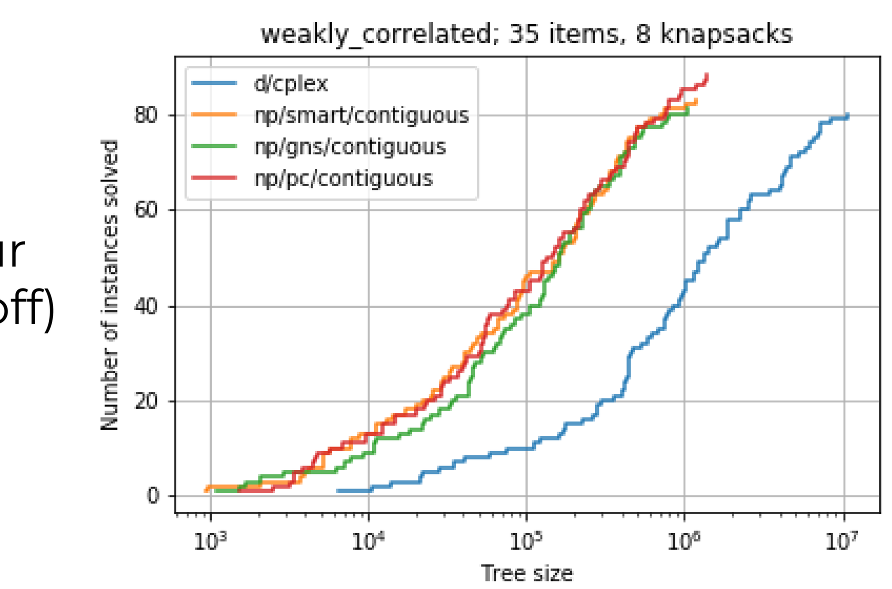
Cover cut generation: cheap, easy to generate families (instead of solving NP-hard separation problem).

$10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 6x_6 + 5x_7 + 4x_8 \le 26$

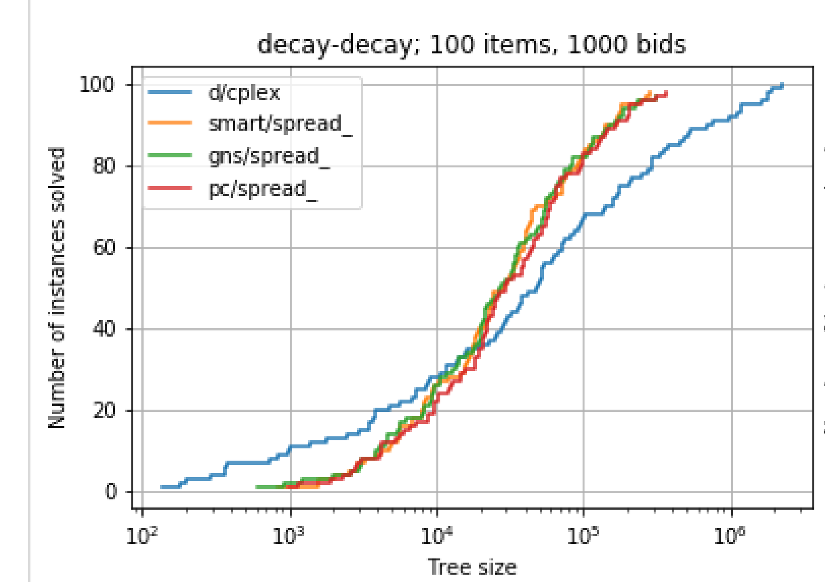
Contiguous covers	Spread covers
$\{1,2,3\}, \{2,3,4,5\}, \{3,4,5,6\}, \{4,5,6,7,8\}$	$\{1,5,6,7\}, \{2,5,6,7,8\}, \{3,5,6,7,8\}, \{4,5,6,7,8\}$

Branch-and-Cut integration: at each node, add the 10 deepest lifted cuts that separate x^{LP}.

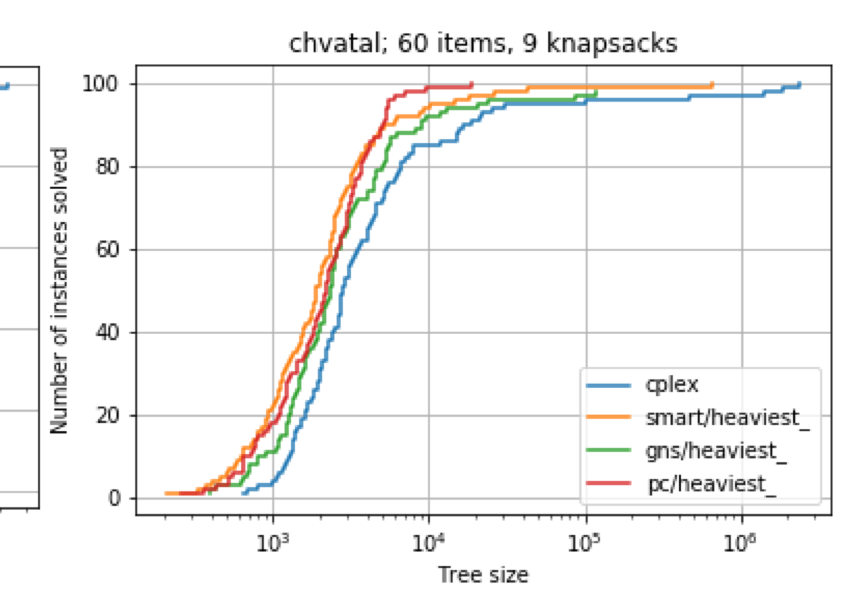
Default CPLEX B&C vs our methods (only presolve off)



Default CPLEX B&C vs our methods (everything off)



Bare-bones CPLEX B&C vs our methods (everything off)



Practical Issues and Future Research

- Smaller trees didn't translate to run-time improvements, though often we weren't much slower and sometimes we were faster
- More comprehensive suite of experiments needed to see where PC lifting shines.
- Further investigation of desirable numerical properties (half-integral coefficients) of PC lifting.

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