Warm Starting of Mixed Integer Linear Optimization Problems via Parametric Disjunctive Cuts

Shannon Kelley[1]; Aleksandr M. Kazachkov[2]; Ted Ralphs[1]
[1]Lehigh University, [2]University of Florida

Can we parameterize disjunctive cuts to improve solver performance for a sequence of MILPs? If so, how?

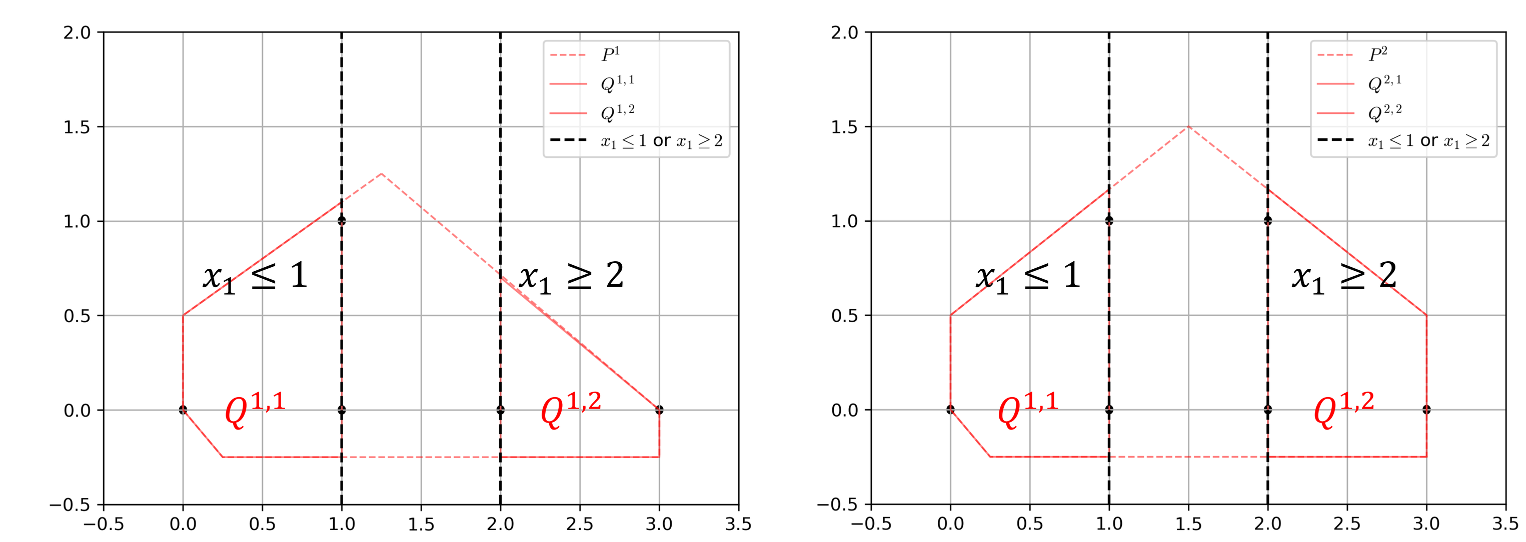
THEORY

Input: A sequence of mixed integer linear optimization problems (MILPs), $\{IP_1, \dots, IP_K\}$, sharing same variables.

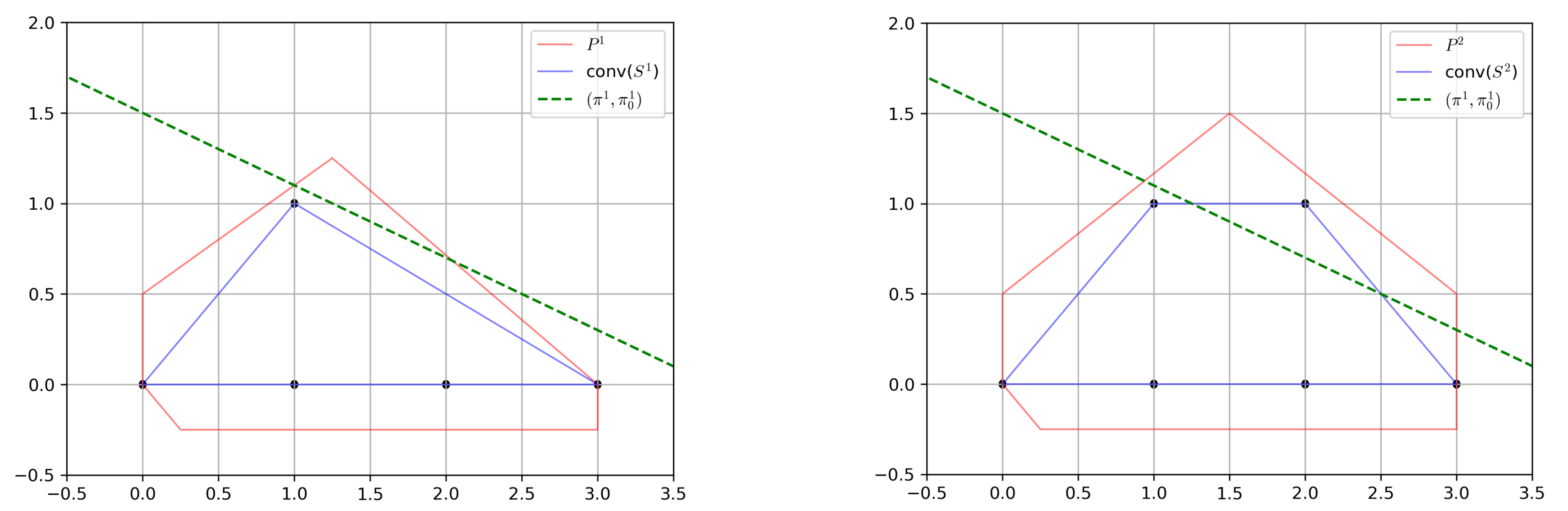
$$(IP_k)\left[\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^k x \\ & \left.A^k x \geq b^k\right] (P^k) \\ & x_j \in \mathbb{Z} \quad \text{for } j \in I \end{array}\right] (S^k)$$

Let $\{X^t\}_{t\in[T]}$ be a **disjunction**, where $X^t := \{x \in \mathbb{R}^n : D^t \geq D_0^t\}$. $\{X^t\}_{t\in[T]}$ is **valid** for a set $S \subseteq \mathbb{R}^n$ if $S \subseteq \bigcup_{t=1}^{T} X^t$. Let $Q^{kt} := P^k \cap X^t$.

Opportunity: If the sequence varies little, a MILP solver might employ similar disjunctions in solving each instance.



The same disjunction yields pairs of child problems that Branch-and-Bound might also generate for their respective instances.



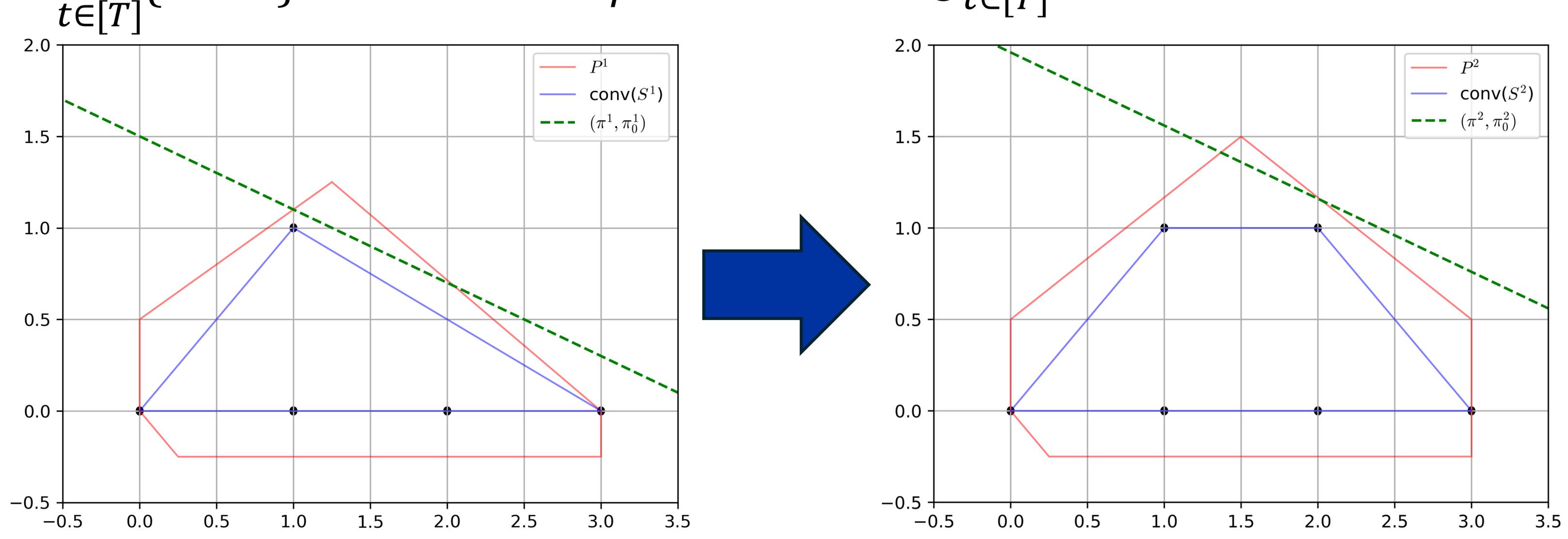
(π^1, π_0^1) is a valid VPC for IP_1, but it is not valid for IP_2 when applied directly.

Lemma 1: Let (π, π_0) be a valid cut for IP_k. Let $A^{kt} := \begin{bmatrix} A^k \\ D^t \end{bmatrix}$ and $b^{kt} := \begin{bmatrix} b^k \\ D_0^t \end{bmatrix}$. Then there exists v^t such that

$$\left.\begin{array}{l} \pi^T = v^t A^{kt} \\ \pi_0 \leq v^t b^{kt} \\ v^t \geq 0 \end{array}\right\} \text{ for all } t \in [T]$$

We refer to $\{v^t\}_{t\in[T]}$ as **Farkas multipliers.**

Lemma 2: Let $\{v^t\}_{t\in[T]}$ be a set of nonnegative Farkas multipliers for a disjunction $\{X^t\}_{t\in[T]}$. For $\ell \in [K]$ and for all $j \in [n]$, let $\alpha_j := \max\limits_{t\in[T]}\{v^t A_{.j}^{\ell t}\}$ and $\beta := \min\limits_{t\in[T]}\{v^t b^{\ell t}\}$. Then $\alpha^T x \geq \beta$ is valid for $\bigcup_{t\in[T]} X^t$.



Parameterizing (π^1, π_0^1) yields (π^2, π_0^2), a valid disjunctive cut for IP_2.

Idea: Generate $\mathcal{V}$-Polyhedral Disjunctive Cuts (VPCs) via [1] for some instances and reapply them to the remaining instances.

Problem: VPCs can become invalid when constraints are perturbed.

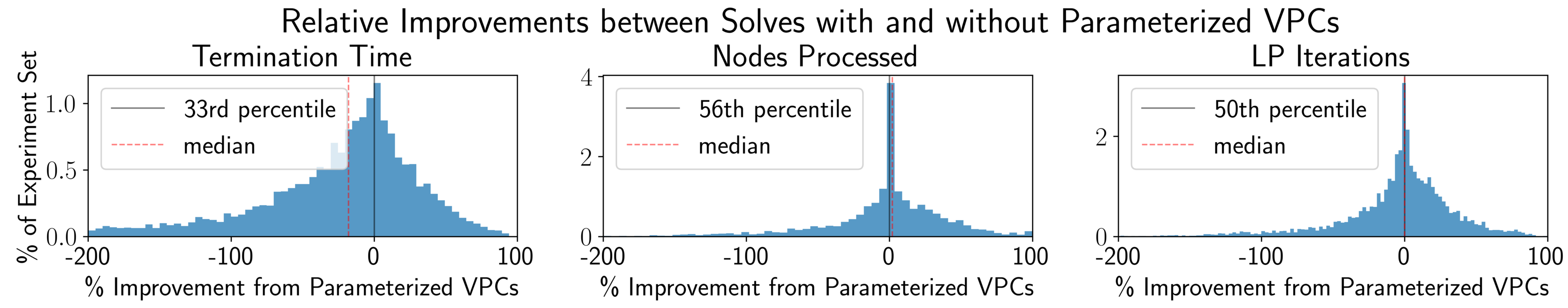
Solution: After generating a VPC for IP_k, parameterize it to ensure its validity when applied to IP_ℓ for $\ell \geq k$.

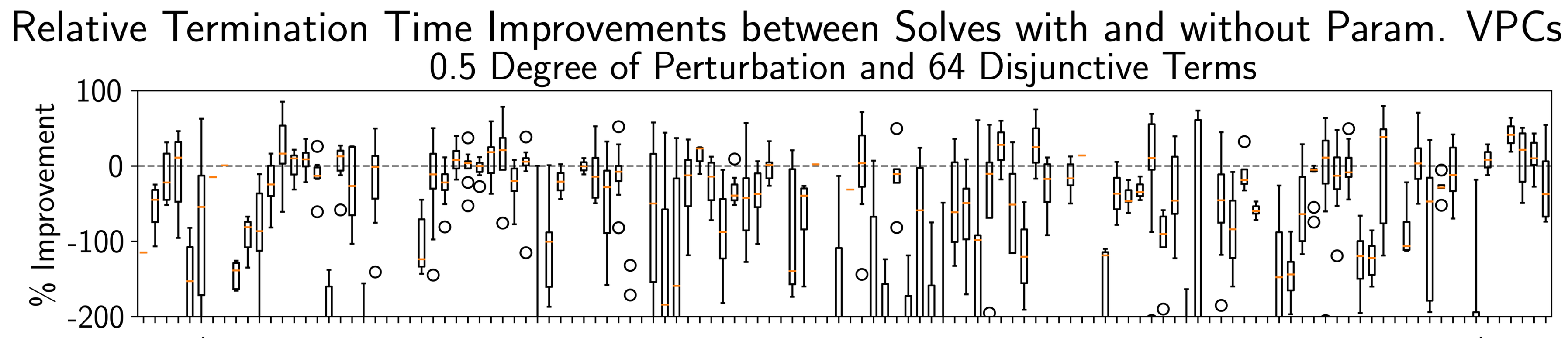
COMPUTATION

Experimental Setup:

- The Base Set consists of 104 presolved MIPLIB 2017 instances with at most 5000 variables and 5000 constraints.
- The Experiment Set consists of 5 random perturbations of objective, RHS, and/or matrix for each instance in Base Set.
- Replications vary by the following parameters:
  - 4, 16, or 64 term disjunctions for VPC generation
  - 0.5 or 2 degrees of random perturbation
  - No VPCs, VPCs via [1], or parameterized VPCs
- The Experiment Set is solved for each combination of parameters using Gurobi 10.

		Average Root Optimality Gap Closed			Average Root Node Processing Time			Average % Perturbed
Degree	Terms	No VPCs	VPCs via [1]	Param. VPCs	No VPCs	VPCs via [1]	Param. VPCs	Terms Becoming Feasible
0.5	4	61.87%	62.35%	62.30%	0.929	10.480	0.999	0.000%
	16	61.87%	62.96%	62.82%	0.936	29.483	1.394	0.102%
	64	61.87%	63.55%	63.35%	0.921	56.614	2.185	0.201%
2	4	63.46%	63.45%	63.36%	0.892	4.293	0.927	0.000%
	16	63.46%	63.76%	63.53%	0.870	17.576	1.394	0.558%
	64	63.46%	64.73%	63.91%	0.861	48.773	2.295	0.596%





(Each set of box and whiskers represents the perturbations of one presolved MIPLIB 2017 Instance)

Yes, we can! For IP_k, find disjunctive cuts and their Farkas multipliers. For IP_ℓ with $\ell \geq k$, use Farkas multipliers to compute new valid inequalities.

CONCLUSION

Key Takeaways:

- Parameterization amortizes the cost of generating VPCs via [1], often still improving the strength of default cuts at the root.
- A significant number of perturbed instances see improvements to run time, nodes processed, and LP iterations.

Next Steps:

- Generalize [1] to include infeasible disjunctive terms. Currently, $v^t := 0$ for IP_k generating VPCs via [1] and t such that $Q^{kt} = \emptyset$. For $\ell \geq k$ and $Q^{\ell t} \neq \emptyset$, this weakens parameterization occurring from Lemma 2.
- Better understand why parameterized VPCs help for some perturbations of the same degree and base instance but not others.

References:
[1] Egon Balas and Aleksandr M. Kazachkov. $\mathcal{V}$-polyhedral disjunctive cuts, 2022.
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[3] Julius Farkas. Theorie der einfachen Ungleichungen. J. Reine Angew. Math., 124:1–27, 1902