Accurate Linear Cutting-Plane Relaxations for ACOPF

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ACOPF

- Objective: $\sum_{k \in \mathcal{G}} F_k(P_k^g)$
- Active and reactive power flow definitions:

$$P_{km} = G_{kk}{v_k}^2 + G_{km}v_k v_m \cos(\theta_k - \theta_m) + B_{km}v_k v_m \sin(\theta_k - \theta_m)$$
$$Q_{km} = -B_{kk}{v_k}^2 + B_{km}v_k v_m \cos(\theta_k - \theta_m) - G_{km}v_k v_m \sin(\theta_k - \theta_m)$$

- Active power balance: $\sum_{\{k,m\} \in \delta(k)} P_{km} = P_k^g - P_k^d$
- Reactive power balance: $\sum_{\{k,m\} \in \delta(k)} Q_{km} = Q_k^g - Q_k^d$
- Line capacity: ${P_{km}}^2 + {Q_{km}}^2 \leq U_{km}$
- Voltage bound: $(V_k^{\min})^2 \leq {v_k}^2 \leq (V_k^{max})^2$

A tight but very difficult SOC relaxation

- $v_k^{(2)} := v_k^2, \qquad c_{km} := v_k v_m \cos(\theta_k - \theta_m), \qquad s_{km} := v_k v_m \sin(\theta_k - \theta_m)$
- Linearized active and reactive power flows:

$$P_{km} = G_{kk}v_k^{(2)} + G_{km}c_{km} + B_{km}s_{km}$$
$$Q_{km} = -B_{kk}v_k^{(2)} + B_{km}c_{km} - G_{km}s_{km}$$

- Jabr rotated-cone inequality: ${c_{km}}^2 + {s_{km}}^2 \leq v_k^{(2)} v_m^{(2)}$
- i2 definition: $i_{km}^{(2)} := \alpha_{km}v_k^{(2)} + \beta_{km}v_m^{(2)} + \gamma_{km}c_{km} + \zeta_{km}s_{km}$
- i2 rotated-cone inequality: ${P_{km}}^2 + {Q_{km}}^2 \leq v_k^{(2)} i_{km}^{(2)}$
- Recall:

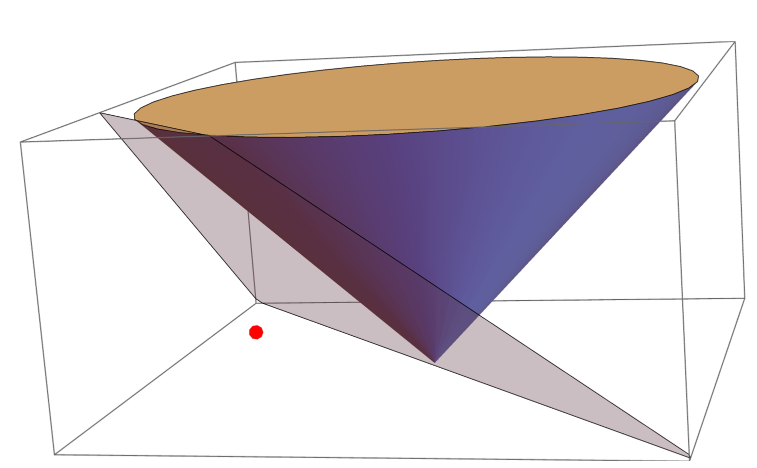
$$x^2 + y^2 \leq wz \iff \left\| \begin{pmatrix} 2x \\ 2y \\ w - z \end{pmatrix} \right\|_2 \leq w + z$$

From SOCs to LPs

- Cuts are **sparse** and **cheap**: $(\bar{x}, \bar{t}) \notin \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ : ||x||_2 \leq t\}$ then

Outer-envelope cut $\quad \bar{x}^\top x \leq ||\bar{x}||_2 t$

- **Warm-starts**: "We don't start from scratch" and structural inequalities

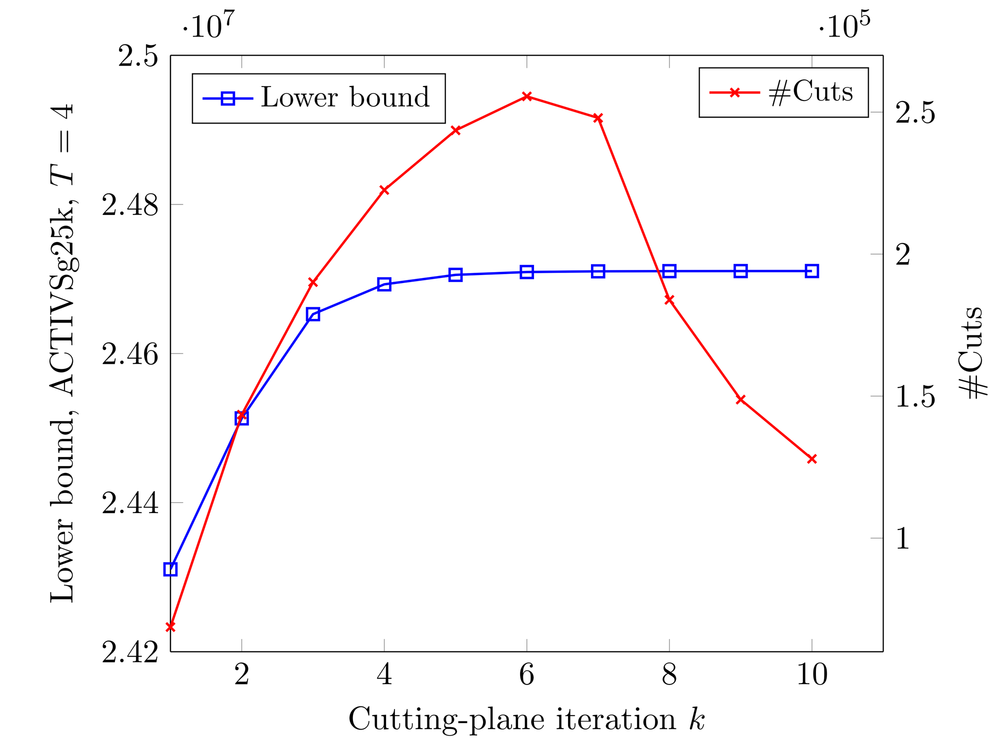


Cutting-Plane Algorithm

$M_0 \leftarrow$ SOC Relaxation with only linear constraints

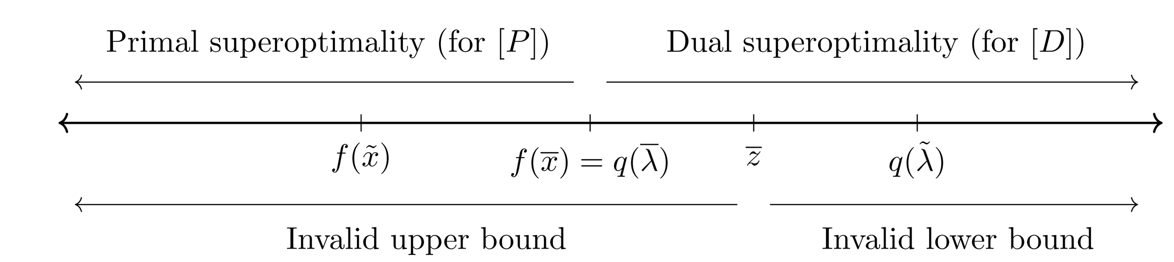
Algorithm 1 Cutting-Plane

Initialize $M \leftarrow M_0$
while $t < T$ **do**
  $\bar{x} \leftarrow \text{argmin} M$
  Compute outer-envelope cuts for top p_i% violated branches above threshold ϵ_i
  Cut Mgmt: Add cuts to M if they are not ρ_i-parallel to cuts in M
  Cut Mgmt: Drop cuts of age $\geq T_{age}$ whose slack is ϵ_i
end while



Accuracy of Lower Bounds

- Minimization problem $[Z]$ with optimal value $\bar{z}$
- Convex relaxation $[P]$ of $[Z]$ and its dual $[D]$ (assume strong-duality)
- ϵ-feasible $(\tilde{x}, \tilde{\lambda})$ and optimal $(\bar{x}, \bar{\lambda})$ primal-dual pairs for $[P]$-$[D]$
- Robust guarantees for LPs and convex QPs



$T = 4$	Cutting-Plane														i2 SOCP+								ACOPF
				First Iteration				Last Iteration				Total				Obj			Time			
Case	#Vars	#Cons	FTime	Obj	Time	#Cuts	DInfs	Obj	Time	#Cuts	DInfs	Time	Rnds	#Vars	#Cons	Gurobi	Knitro	Mosek	Gurobi	Knitro	Mosek	PBound
9241pegase	524028	538308	11.36	1258422.42	48.99	68472	3.61e-06	1270138.75	100.47	189019	5.18e-08	1268.90	7	451546	643811	-	-	-	411.55	TLim	92.33	1297205.96
ACTIVSg10k	456684	490360	10.27	10117471.23	30.65	28184	5.72e-09	10222978.04	23.94	51188	5.01e-08	306.72	10	364824	437972	-	-	-	243.69	TLim	79.58	10265810.55
ACTIVSg25k	1145783	1212392	25.46	24310520.83	93.95	68676	1.01e-07	24710662.67	92.3	127912	1.61e-08	1206.14	9	961854	1387817	-	-	-	589.52	TLim	200.22	24835784.33
30000goc-api	1253699	1366264	30.24	5185673.79	97.36	59732	3.54e-07	5706550.06	80.15	105806	9.79e-07	1146.25	12	1018714	1524129	-	-	-	1761.14	TLim	286.2	6563873.25
30000goc-sad	1253699	1366264	28.03	4316717.55	138.08	179704	7.84e-07	4341119.26	77.37	91071	1.06e-06	994.70	9	1018714	1524129	-	-	-	727.32	TLim	247.6	-
ACTIVSg70k	3115935	3316972	68.7	67721489.41	350.58	258228	3.04e-08	69422696.57	369.05	597271	1.11e-07	1301.25	3	2584042	3765747	-	-	-	1312.16	TLim	405.88	70148300.24
78484epigrids-api	4179320	4225032	96.42	60947456.94	494.72	625780	1.53e-04	61271585.98	563.69	1047809	1.52e-04	1523.35	2	3558322	4997073	-	-	-	TLim	TLim	1117.23	62011780.96
78484epigrids-sad	4179320	4225032	97.61	58930587.37	743.6	888916	1.20e-04	59051305.24	845.85	1133203	1.20e-04	1569.46	1	3558322	4997073	-	-	-	TLim	TLim	1117.35	-

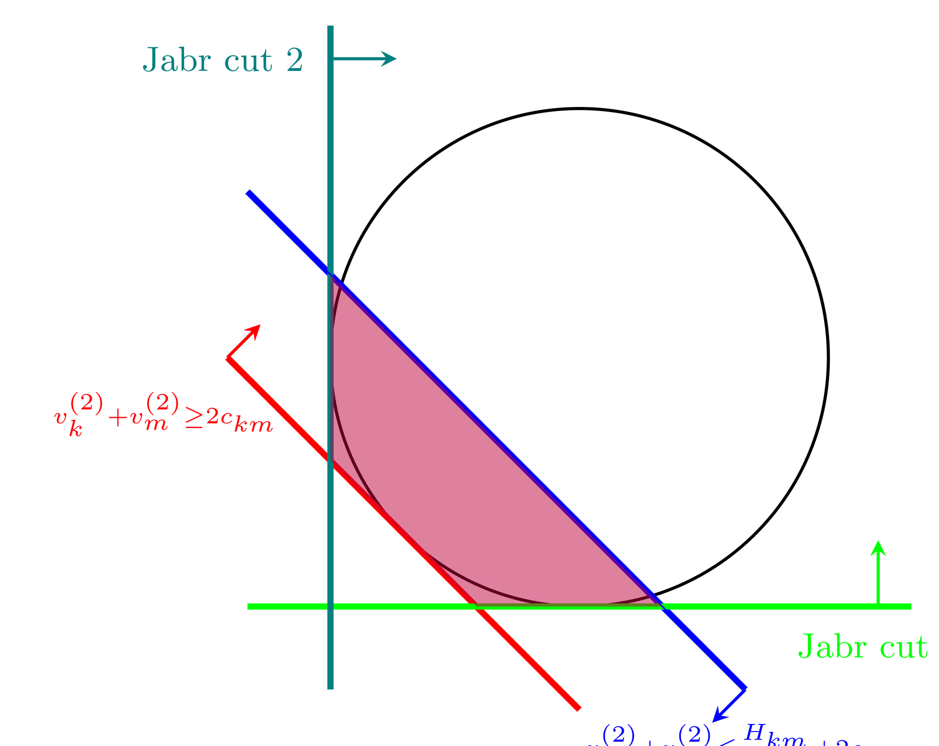
$T = 24$	Cutting-Plane													i2 SOCP+								DCOPF	
			First Iteration				Last Iteration				Total				Obj			Time				
Case	#Vars	#Cons	Obj	Time	#Cuts	DInfs	Obj	Time	#Cuts	DInfs	Time	Rnds	#Vars	#Cons	Gurobi	Knitro	Mosek	Gurobi	Knitro	Mosek	Obj	Time
9241pegase	3151388	3258748	6865741.86	368.23	410832	4.33e-08	6925029.02	2427.02	1354277	1.25e-03	8419.66	4	2709266	3957551	-	-	-	1972.91	TLim	525.04	7013508.62	36.02
ACTIVSg10k	2752524	2991860	54565119.63	226.82	169104	7.5e-09	55184650.09	211.53	298812	7.18e-08	2869.25	11	2302130	3446078	-	-	-	3918.14	TLim	396.91	54241448.25	37.77
ACTIVSg25k	6898863	7371032	132201219.46	816.18	412056	4.74e-07	134142620.00	2042.59	1385811	2.08e-06	6991.16	3	5771114	8536377	-	-	-	4451.39	TLim	963.91	131325272.36	95.18
30000goc-api	7539819	8268104	28586461.36	755.74	358392	1.7313e-06	31616476.13	884.66	1064587	2.06e-05	6799.92	7	6112274	9356989	-	-	-	TLim	TLim	1139.21	171.42	-
30000goc-sad	7539819	8268104	25717079.58	1091.27	1078224	2.52e-06	25811075.96	1021.24	1393213	1.23e-05	6249.53	5	6112274	9356989	-	-	-	3213.23	TLim	1125.14	25279287.29	108.86
ACTIVSg70k	18747555	20109632	350310736.78	2641.18	1549368	2.407e-07	357083446.24	2687.31	3359811	2.98e-08	8496.71	2	1550424	23139407	-	-	-	5061.29	TLim	1331.11	-	829.66
78484epigrids-api	25110280	25487652	-	1521.64	3754680	-	-	1521.64	3754680	-	2138.78	0	21349922	30681233	-	-	-	TLim	287.62	1434.04	INF	380.67
78484epigrids-sad	25777488	24855336	344805634.00	4749.24	5333496	1.34e-04	-	TLim	6915253	-	11561.02	1	21349922	30681233	-	-	-	TLim	284.28	1448.87	-	513.08

Cutting-planes and Power Losses

- Power loss at line $\{k,m\}$: $\ell_{km} := P_{km} + P_{mk}$.
- **Structural result "There is no free generation"**: $\ell_{km} \geq 0$.
- For any $\lambda \in \mathbb{R}^3$, $||\lambda||_2 = 1$, we have the linear outer-envelope cut

$$\lambda_1(2c_{km}) + \lambda_2(2s_{km}) + \lambda_3(v_k^{(2)} - v_m^{(2)}) \leq ||\lambda||_2\, ||(2c_{km}, 2s_{km}, v_k^{(2)} - v_m^{(2)})^\top||_2 \leq v_k^{(2)} + v_m^{(2)}$$

- For example, if line $\{k,m\}$ is "simple", then $\ell_{km} = G_{kk}(v_k^{(2)} + v_m^{(2)} - 2c_{km})$
- Hence the Jabr cut $\lambda = (1,0,0)$ implies $\ell_{km} \geq 0$



Numerical Stability

- **Bad i2s**: $\alpha_{km} \gg 1$

$$v_k^{(2)} + \frac{\beta_{km}}{\alpha_{km}}v_m^{(2)} + \frac{\gamma_{km}}{\alpha_{km}}c_{km} + \frac{\zeta_{km}}{\alpha_{km}}s_{km} \geq 0$$
$$v_k^{(2)} + \frac{\beta_{km}}{\alpha_{km}}v_m^{(2)} + \frac{\gamma_{km}}{\alpha_{km}}c_{km} + \frac{\zeta_{km}}{\alpha_{km}}s_{km} \leq \frac{H_{km}}{\alpha_{km}}$$

where H_{km} is an upper bound for $i_{km}^{(2)}$