Benders cuts via corner polyhedra: an application to the stochastic vehicle routing problem

Mathes Jun Ota*, Riccardo Fukasawa, Aleksandr M. Kazachkov

Institution: University of Waterloo, Department of Combinatorics and Optimization; Email: mjota@uwaterloo.ca

Introduction

Consider the following mixed integer linear program. (MILP) min \( c^\top x + d^\top y \)

s.t. \( TX + Qy = h \),

\( Ax \geq b \),

\( Gy = g \),

\( x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \),

\( y \in \mathbb{R}^{m_1} \).

Benders decomposition: “project out” the \( y \)-variables.

Master problem: \( \min \{ \theta^\top c \} \)

\( \begin{aligned} & s.t. \quad x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \\ & \theta \geq 0 \end{aligned} \)

\( \text{(Benders cuts).} \)

Preliminaries

Definition 2 (support): For our purposes, for any set of vectors \( P \), we refer to the support of \( P \) as the function \( \sigma_P : s \rightarrow \text{supp}(P \cdot s) \).  

Fact 1 (cone optimality): Let \( y^* \) be an optimal BFS for \( \min_{x \in X} \{ x^\top y \} \) with basis \( B \). Let \( \{e_r\}_{r \in B} \) be the rays associated with the non-basic variables w.r.t. \( B \). Define \( C(y^*, R) = \{ y^* \} + \text{cone}(R) \), then \( \sigma(y) = \sigma(y^*, R) \).

Lemma 1 [DFK24]: For every \( a \in \mathbb{R}^m \) and every set of points \( P \subseteq \mathbb{R}^{m_1} \), we have that \( \sigma(d + Q^\top a) = \sigma_{M(Y)}(1, a) \).

Separating stronger Benders cuts via projected corner polyhedra

An initial idea: Use the Lagrangian dual to make use of the oracle \( O \).

\[ f(x) = \max_{z \in \mathbb{R}^n} \left\{ \sum_{i=1}^m d_i z_i - \sum_{i=1}^m Q_{i,j} y_i \right\} = \max_{\alpha \in \mathbb{R}^d} \left\{ -\sum_{i=1}^m d_i \alpha_i + (Q_{i,i})^\top \alpha \right\} \]

\( (1) \)

This gives a Benders optimality cut: for all \( \alpha \in \mathbb{R}^d \),

\[ \begin{aligned} \theta^\top \alpha &\leq \theta^\top \alpha^* \iff \theta^\top \alpha \leq \theta^\top \alpha^* \leq \theta^\top \alpha \end{aligned} \]

\[ \geq \sigma_{M(Y)}(1, a) \]  

(2)

However, in our preliminary experiments (using the subgradient/volume method), this approach still leads to weak cuts with bad convergence. So we need to do better...

Theorem 1 [DFK24]: Let \( (\theta^*, x^*) \) be a candidate solution (see Figure 1) and let \( \theta^* \) be such that \( (\theta^*, x^*) \) violates the corresponding inequality (2). Let \( C = C(y^*, R) \) be an optimal cone w.r.t. \( \sigma(d + Q^\top a) \), then \( \theta^* \geq M(C) \) and \( \theta^* \geq M(Y ) \).

Proof: We show that \( (\theta^*, x^*) \) is valid for \( M(C) : \sigma(d + Q^\top a) \) and \( \sigma(d + Q^\top a) \). \( \theta^* \geq M(Y ) \).

Computational experiments

Standard Benders based on LP duality (Figure 1) and Lagrangian duality (Equation (1)) cannot solve root node in 30 minutes.  

Algorithms: ILS: our implementation of a state-of-the-art integer L-shaped (ILS) algorithm [2]; +Con: addition of our cuts (Theorem 2).

For gap, we ignored instances that an algorithm was unable to find a feasible solution within the time limit.

For times, the numbers outside parentheses are averages over all instances; numbers inside parentheses ignore instances that were not solved within the time limit.

Table 1: Computational experiments on instances from Jabali et al. (2014)

Root Gap | Time (s) | Solved
--- | --- | ---
40 4 4.3% | 1.8% | 208 (931) | 59 (522) | 9 / 10 | 10 / 10
50 4 3.2% | 1.7% | 406 (232) | 482 (299) | 8 / 10 | 8 / 10
60 4 5.4% | 2.4% | 913 (319) | 806 (271) | 6 / 10 | 7 / 10
70 4 6.5% | 2.4% | 1362 (326) | 1061 (203) | 3 / 10 | 6 / 10

References