Strong Formulation of Hybrid Control Problem with Tridiagonal Inverse Matrix

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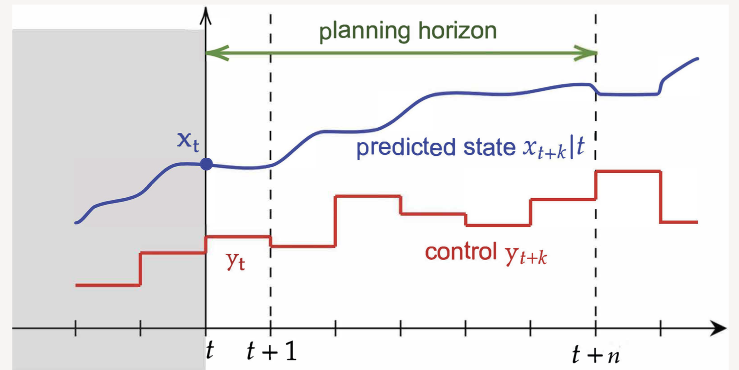
Hybrid Model Predictive Control (HMPC)

Hybrid System: Discrete & continuous dynamics interact through mode switch [2].

- Binary variable $z_t \in \{0,1\}^d$: mode at time t
- In each mode, system control is a continuous optimization.

Model Predictive Control: Approach for long-term control problems with uncertainties.
→ Iterative finite-horizon control optimization for real-time decisions.

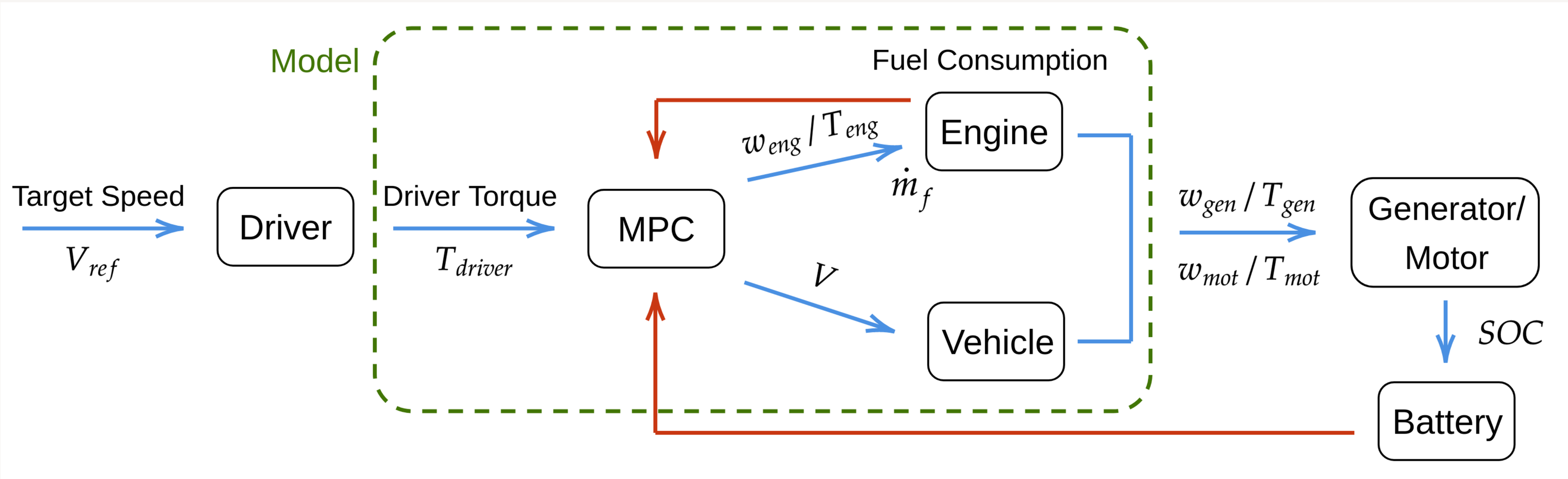


At each period t,
1. Measure current state x_t
2. Solve n-period HCP with initial state x_t
3. Implement first control y_t
4. $t \leftarrow t+1$ and repeat

MIQP Formulation of n-period Hybrid Control Problem (HCP): For $Q_t \succ 0, \forall t \in [n]$

$$
\begin{aligned}
(n-\text{HPC}) \quad & \min_{x,y,z} \frac{1}{2}\sum_{t=1}^{n+1} (x_t - r_t)^\top Q_t (x_t - r_t) + \sum_{t=1}^{n} p_t z_t \\
\text{s.t.} \quad & x_1 = x_0, \\
& x_{t+1} = A_t x_t + B_t y_t + c_t z_t + d_t, && \forall t \in [n] \\
& l_{y_t} z_t \le y_t \le u_{y_t} z_t, && \forall t \in [n] \\
& l_{x_t} \le x_t \le u_{x_t}, && \forall t \in [n+1] \\
& x \in \mathbb{R}^{(n+1)\times d_x},\ y \in \mathbb{R}^{n \times d_y},\ z \in \{0,1\}^n
\end{aligned} \tag{1}
$$

Application: Energy management of power-split HEV



State, control, measured disturbance:

$$
x = \begin{bmatrix} SOC \\ \dot{m}_f \end{bmatrix} \quad y = \begin{bmatrix} V \\ w_{\text{eng}} \\ T_{\text{eng}} \end{bmatrix} \quad v = \begin{bmatrix} V^{\text{ref}} \\ T_{\text{driver}} \end{bmatrix}
$$

Control system with 2 modes, engine on/off [1]

$$
x(k+1) = A_i x(k) + B_i y(k) + D_i v(k) + F_i, \qquad i = 0(\text{off}), 1(\text{on})
$$

References

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[2] F. Borrelli, A. Bemporad, and M. Morari. *Predictive control for linear and hybrid systems.* Cambridge University Press, 2017.

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Simple Hybrid Control Problem

$$
\begin{aligned}
\min\ & \frac{1}{2}\sum_{t=1}^{n+1}(x_t - r_t)^2 + \sum_{t=1}^{n} p_t z_t \\
\text{s.t.}\ & x_{t+1} = x_t + y_t, && \forall t \in [n] \\
& y_t(1-z_t) = 0, && \forall t \in [n] \\
& z_t \in \{0,1\}, && \forall t \in [n]
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min\ & \sum_{t=1}^{n+1}\left(x_0 + \sum_{s=1}^{t-1} y_s - r_t\right)^2 + \sum_{t=1}^{n} p_t z_t \\
\text{s.t.}\ & y_t(1-z_t) = 0, && \forall t \in [n] \\
& z_t \in \{0,1\}, && \forall t \in [n]
\end{aligned}
$$

$$
\Leftrightarrow \ \min \frac{1}{2} y^\top \tilde{Q} y + \tilde{a}^\top y + p^\top z + \sum_{t=1}^{n+1}(x_0 - r_t)^2 \ \text{ s.t. } y_t(1-z_t) = 0,\ z_t \in \{0,1\},\ \forall t \in [n] \tag{2}
$$

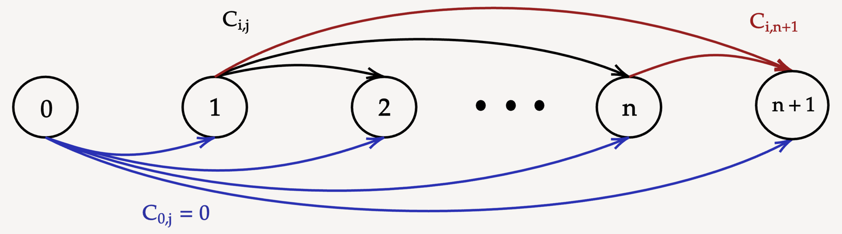
where $\tilde{Q}_S^{-1}$ is tridiagonal and sum of rank-1 matrices, $\forall S = \{\pi_1, \ldots, \pi_k\} \subseteq \{1, \ldots, n\}$ s.t. $\pi_1 < \cdots < \pi_k$.

$$
\tilde{Q}_S = \begin{bmatrix}
n-\pi_1 & n-\pi_2 & n-\pi_3 & \cdots & n-\pi_k \\
n-\pi_1 & n-\pi_2 & n-\pi_3 & \cdots & n-\pi_k \\
n-\pi_3 & n-\pi_3 & n-\pi_3 & \cdots & n-\pi_k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n-\pi_k & n-\pi_k & n-\pi_k & \cdots & n-\pi_k
\end{bmatrix} \succ 0 \ \Rightarrow \ \tilde{Q}_S^{-1} = \sum_{i=1}^{k-1} \overbrace{\frac{1}{\pi_i - \pi_{i+1}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{[i,i+1]}}^{C_{\pi_i, \pi_{i+1}}} + \overbrace{\frac{1}{\pi_k}(1)_{[k]}}^{C_{\pi_k, n+1}}
$$

where $(M)_{[I]}$ is a $k \times k$ matrix with M is the submatrix defined by indices I and all other elements are zero.

Consider a directed acyclic graph $G = (V, A)$

- $V = \{0, 1, \ldots, n, n+1\}$
- $A = \{(i, j) : 0 \le i < j \le n+1\}$



Let $P(G)$ be a set of $0-(n+1)$ paths in G and

$$
\mathcal{P} = \left\{ (z, W, \delta) : \delta \in P(G),\ W = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} C_{i,j}\delta_{i,j} + \sum_{i=1}^{n} C_{i,n+1}\delta_{i,n+1},\ \ z_j = \sum_{i=0}^{j-1} \delta_{i,j},\ \forall j \in [n] \right\}
$$

Then, (2) can be reformulated as

$$
\begin{aligned}
\min_{\tau, y, z, W, \delta}\ & \frac{1}{2}\tau + \tilde{a}^\top y + p^\top z \\
\text{s.t.}\ & \begin{pmatrix} W & y \\ y^\top & \tau \end{pmatrix} \succeq 0 \\
& (z, W, \delta) \in \mathcal{P}
\end{aligned} \tag{3}
\quad \overset{(\star)}{\Leftrightarrow} \quad
\begin{aligned}
\min_{z, W, \delta}\ & -\frac{1}{2}\tilde{a}^\top W \tilde{a} + p^\top z \\
\text{s.t.}\ & (z, W, \delta) \in \mathcal{P}
\end{aligned} \tag{4}
$$

where $(\star)$ holds as $y^* = -W^* a$ and $z \in \{0,1\}^n$ can be relaxed since $\exists (z^*, W^*, \delta^*) \in \text{ext}(\mathcal{P})$ [3].

Proposition 1. Dynamic Programming

(4) can be solved in $O(n^2)$ as a shortest path problem (SPP) on graph G.

Proposition 2. SOCP Reformulation

There is an SOCP formulation of (3) with $O(n^2)$ conic constraints.

General 1-Dimensional Hybrid Control Problem

The SPP and SOCP reformulations can be applied to general 1-dimensional time-variant HCPs for $q_t > 0$.

$$
\min \frac{1}{2}\sum_{t=1}^{n+1} q_t (x_t - r_t)^2 + p^\top z \ \text{ s.t. } x_{t+1} = \alpha_t x_t + \beta_t y_t,\ y_t(1-z_t) = 0,\ z_t \in \{0,1\},\ \forall t \in [n]
$$

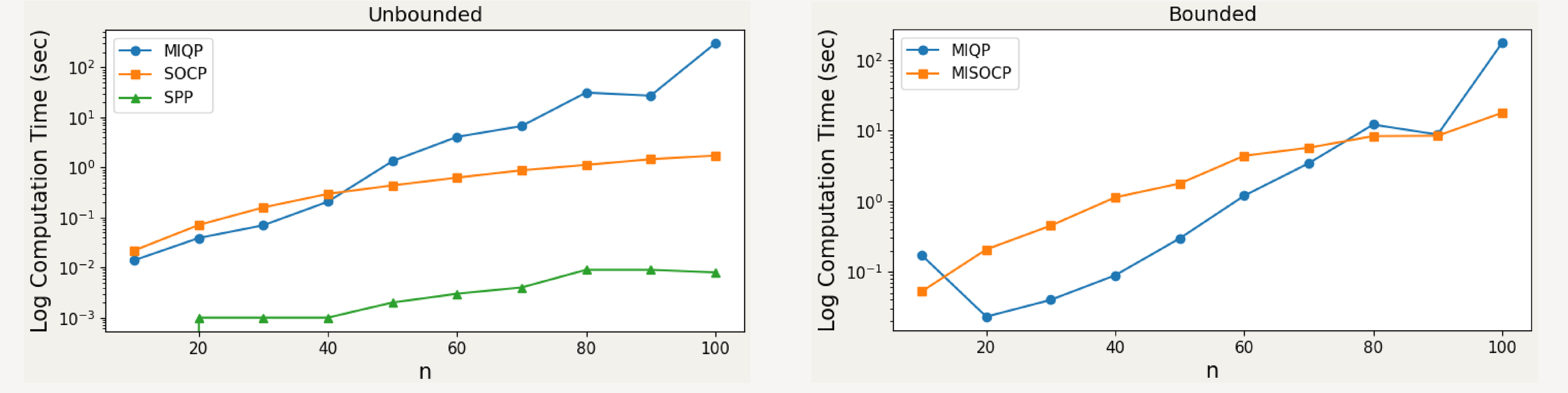
Multi-Dimensional Hybrid Control Problem (1)

- $\tilde{Q}_S^{-1}$ is block tridiagonal, $\forall S \subseteq [n]$.
- Unbounded (1) can be reformulated as SPP/SOCP if $Q \succ 0$ (nec.) & B_t's are full row rank (suff.).
- (1) can be solved using the SOCP formulation with bounds.

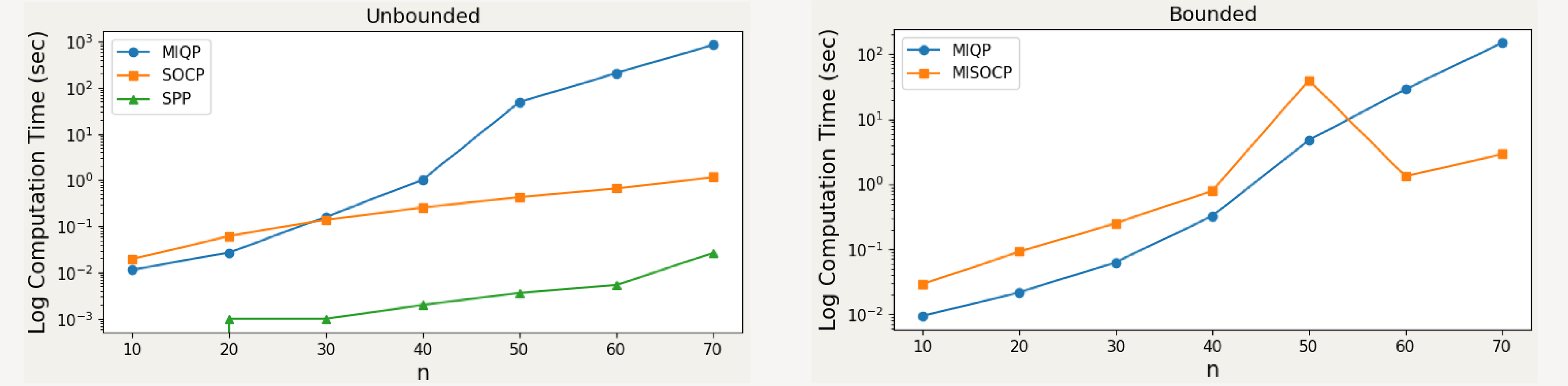
Experiments

Three types of synthetic data tested using Gurobi solver 9.0 (Avg. of 10 instances reported)

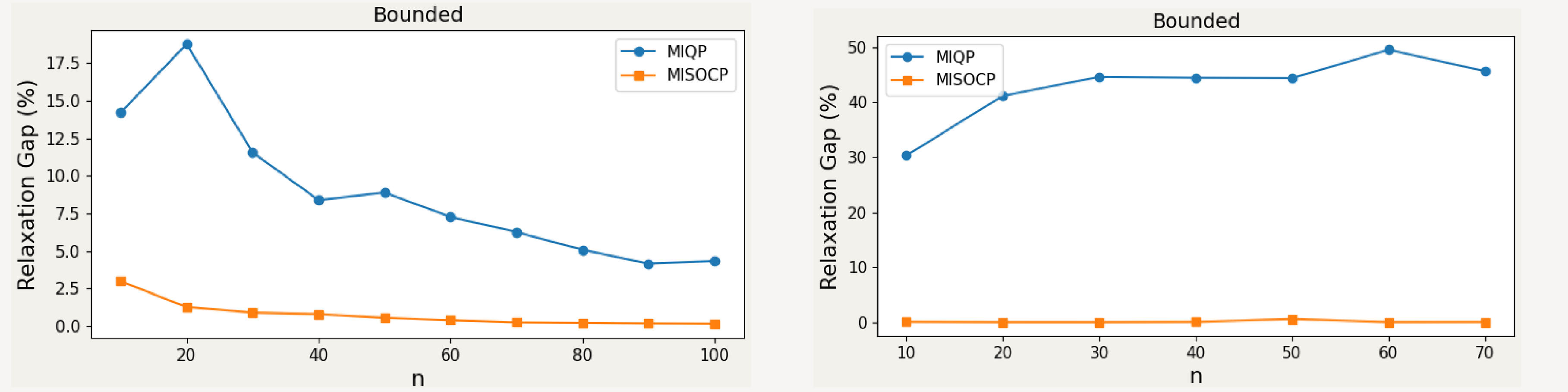
- Type 1: Time-varying cost q_t with linear system $x_{t+1} = x_t + y_t$



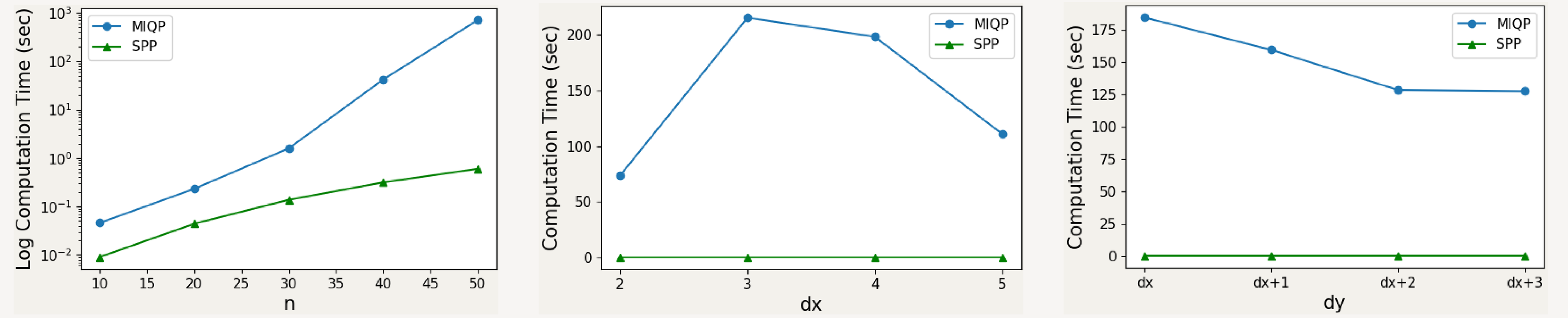
- Type 2: Fixed cost $q_t = q$ with linear system $x_{t+1} = \alpha x_t + \beta y_t$, $\alpha \sim U[1, 1.1]$, $\beta \sim U[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]$



- Relaxation gap of bounded problems of Type 1 and 2



- Type 3. Multi-dimensional HCP with fixed cost Q and linear system $x_{t+1} = Ax_t + By_t + cz_t$



- SOCP of Type 3 is excluded from comparison, as solver had irregular terminations often.
- If $|\alpha| > 1$ (1-dim) or $|\text{eig}(A)| > 1$, computation errors occur frequently for large n.
  → Generate cuts for partial horizon $[t, t+k]$ ($k \ll n$) to form a strong formulation.