Aggregation of Bilinear Bipartite Equality Constraints and Application to FEM Update Problem

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1 Background

Consider a set with two bilinear bipartite equality constraints.

$$S := \left\{ x \in [0,1]^{n_1}, y \in [0,1]^{n_2} \mid x^\top Q_i y + a_i^\top x + b_i^\top y + c_i = 0, \quad i \in [2] \right\}$$

Let us aggregate constraints with weights $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

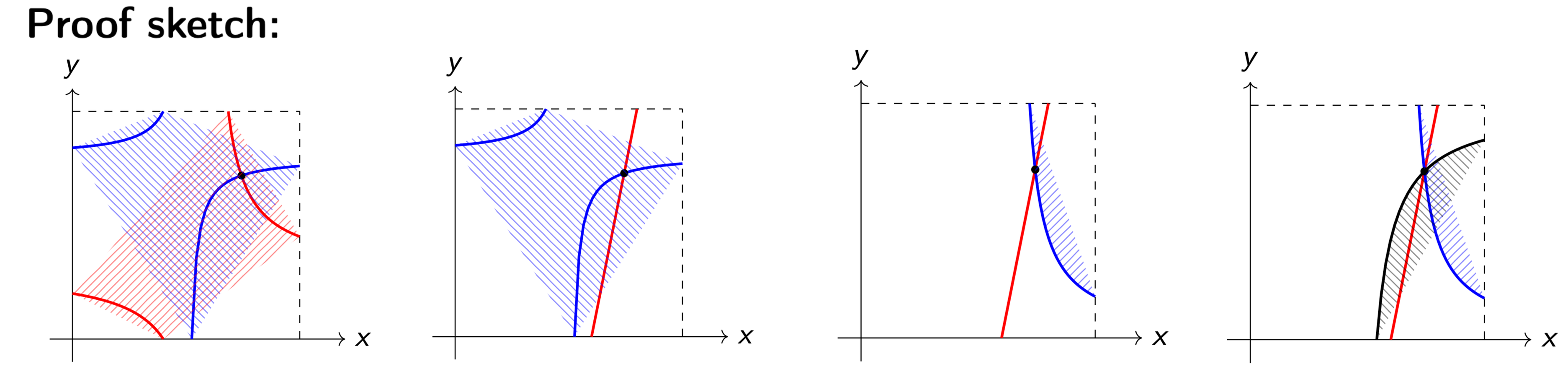
$$S_\lambda := \left\{ x \in [0,1]^{n_1}, y \in [0,1]^{n_2} \;\middle|\; \begin{array}{rl} \lambda_1 \cdot (x^\top Q_1 y + a_1^\top x + b_1^\top y + c_1) & \\ + \lambda_2 \cdot (x^\top Q_2 y + a_2^\top x + b_2^\top y + c_2) & = 0 \end{array} \right\}$$

Remark $S_\lambda \subset S$ for any $\lambda \in \mathbb{R}^2$, hence $\text{conv}(S) \subset \bigcap_{\lambda \in \mathbb{R}^2} \text{conv}(S_\lambda)$.

2 Can conv(S) be represented with intersection of finite number of conv(S_λ)'s?

Proposition 1: Yes, for $n_1 = 1$ and $n_2 = 1$. There is $T \subseteq \mathbb{R}^2$ where $|T| \leq 3$ such that $\text{conv}(S) = \bigcap_{\lambda \in T} \text{conv}(S_\lambda)$.

Proof sketch:



(a) S_1 (blue), S_2 (red), their convex hulls, and feasible region (black) (b) Find a line representation L which gives a same feasible region with S_1 (c) Find $\tilde{S}_1$ intersecting with $[0,1]^2$ box by only one branch (d) Find $\tilde{S}_2$ with $\text{conv}(\tilde{S}_2) \bigcap L$ is on the opposite side of $\text{conv}(\tilde{S}_1) \bigcap L$

Proposition 2: No, for $n_1 = 1$ and $n_2 = 2$, finite number of intersections does not give a convex hull. In other words, for $T \subseteq \mathbb{R}^2$, where $|T| < \infty$,

$$\text{conv}(S) \subsetneq \bigcap_{\lambda \in T} \text{conv}(S_\lambda).$$

Proof sketch: Consider a counterexample:

$$S = \left\{ x, y_1, y_2 \in [0,1]^3 \;\middle|\; \begin{array}{rcl} xy_1 & = & 0.5 \\ xy_2 & = & 0.5 \end{array} \right\}.$$

1. Note that $y_1 = y_2$ for all $(x, y_1, y_2) \in S$, so $\text{conv}(S) \subseteq \{x, y_1, y_2 \in [0,1]^3 \mid y_1 = y_2\}$.
2. $(\frac{3}{4}, \frac{17}{24}, \frac{17}{24}) \in \text{conv}(S)$
3. For any $\lambda \in \mathbb{R}^2$, we can find $\hat{\epsilon}(\lambda) > 0$ such that: $(\frac{3}{4}, \frac{17}{24} + \epsilon, \frac{17}{24} - \epsilon) \in \text{conv}(S_\lambda)$, for all $0 \leq \epsilon \leq \hat{\epsilon}(\lambda)$.
4. Let $\epsilon_0 = \min_{\lambda \in \mathbb{R}^2}\{\hat{\epsilon}(\lambda)\}$, then we found a point that is in all $\text{conv}(S_\lambda)$ but does not satisfy $y_1 = y_2$; hence not in $\text{conv}(S)$.

3 Can conv(S) be represented with intersection of infinite number of conv(S_λ)'s?

Proposition 3: Even an infinite number of intersections does not give a convex hull. In other words,

$$\text{conv}(S) \subsetneq \bigcap_{\lambda \in \mathbb{R}^2} \text{conv}(S_\lambda).$$

Proof sketch: Consider a counterexample:

$$S = \left\{ x_1, x_2, y_1, y_2 \in [0,1] \;\middle|\; \begin{array}{rcl} x_1y_1 - 5x_1y_2 - 2x_2y_1 + 9x_2y_2 & = & 0 \\ 3x_1y_1 + 3x_1y_2 + 5x_2y_1 & = & 6 \end{array} \right\}.$$

1. Note that $\hat{p} = (1, \frac{7}{10}, \frac{7}{8}, \frac{1}{6}) \notin \text{conv}(S)$.
2. When we let $\lambda = (1, \theta)$, for all $\theta \in \mathbb{R}$, we can find $p_1, \ldots, p_4 \in S_\lambda$ and weights $w_1, \ldots, w_4$ such that $\hat{p} = \sum w_i p_i$ and $\sum w_i = 1$.

e.g.,

$$p_1 = \left(1, 0, \tfrac{3\theta+5}{3\theta+1}, 1\right), \quad p_2 = \left(1, 1, \tfrac{6\theta}{8\theta-1}, 0\right), \quad p_3 = \left(1, 1, \tfrac{3\theta-4}{8\theta-1}, 1\right), \quad p_4 = \left(1, \tfrac{3\theta-1}{5\theta-2}, 1, 0\right),$$

$$w_1 = \tfrac{47(1+3\theta)}{120(1+47\theta)}, \quad w_2 = \tfrac{122+1343\theta-1645\theta^2}{120(1+47\theta)(1-2\theta)}, \quad w_3 = \tfrac{799\theta-27}{120(1+47\theta)}, \quad w_4 = \tfrac{-11(1-141\theta)(2-5\theta)}{120(1+47\theta)(1-2\theta)}.$$

4 Can aggregations still be useful?

Remark: Despite results so far, aggregated equalities can provide a tight approximation of conv(S). Random shooting experiment for $n_1 = n_2 = 2$ and minimizing over a random objective function shows significant reduction in gap.

Relative Gap	$\text{conv}(S_1) \cap \text{conv}(S_2)$	$\bigcap_{\lambda \in [-2,2]^2} \text{conv}(S_\lambda)$	$\bigcap_{\lambda \in [-10,10]^2} \text{conv}(S_\lambda)$
Average	5.25%	1.38%	0.56%
Maximum	96.14%	26.38%	22.22%
No. < 0.5%	57%	71%	87%

5 How can we find some "nice" aggregation weights that will give a tight approximation of conv(S)?

Heuristic: When we have a relaxed solution $(\hat{x}, \hat{y})$, let's try to find an aggregation weights that may separate $(\hat{x}, \hat{y})$.

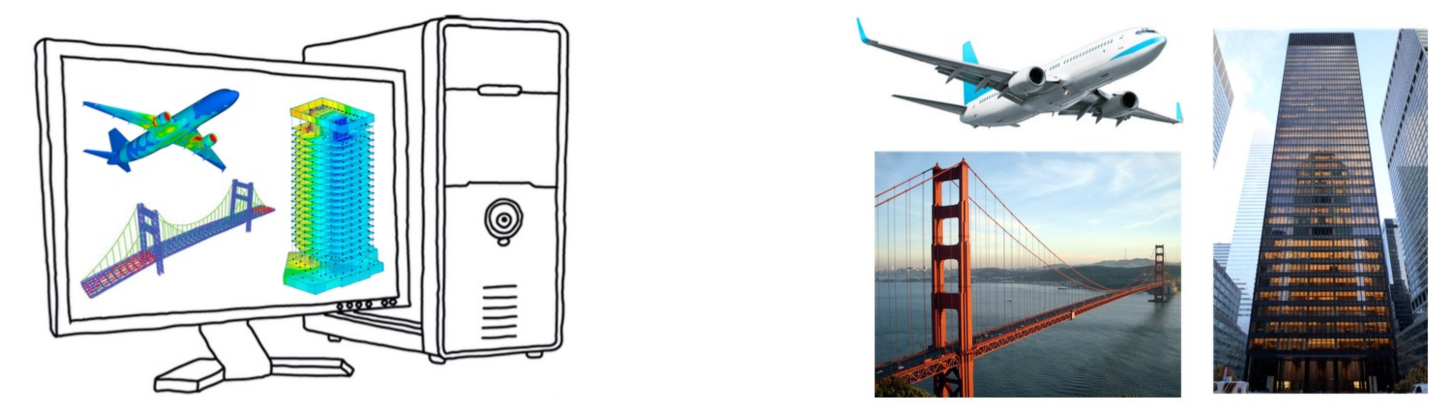
1. Fix y to $\hat{y}$ in S_λ
   $\implies S_\lambda|_{y=\hat{y}}$ is a hyperplane in the x space with parameters defined by $\lambda \in \mathbb{R}^2$.
2. Find $\lambda \in \mathbb{R}^2$ such that the distance between $S_\lambda|_{y=\hat{y}}$ and $\hat{x}$ is maximized. (This is a convex problem and can be solved efficiently.)
3. Go to step 1 and now fix x to $\hat{x}$.
4. Choose among λ's that have maximum distance.

conv(S_λ) separating $(\hat{x}, \hat{y})$
72.00%

6 Application to FEM Update Problem

FEM Update: The finite element (FE) model update problem in structural engineering seeks to minimize the differences between the predicted and actual behaviors of a built structure. This boils down to solving a generalized eigenvalue problem with eigenvalues, eigenvectors and matrix weights being variables, which can then be reformulated as a bilinear bipartite problem.

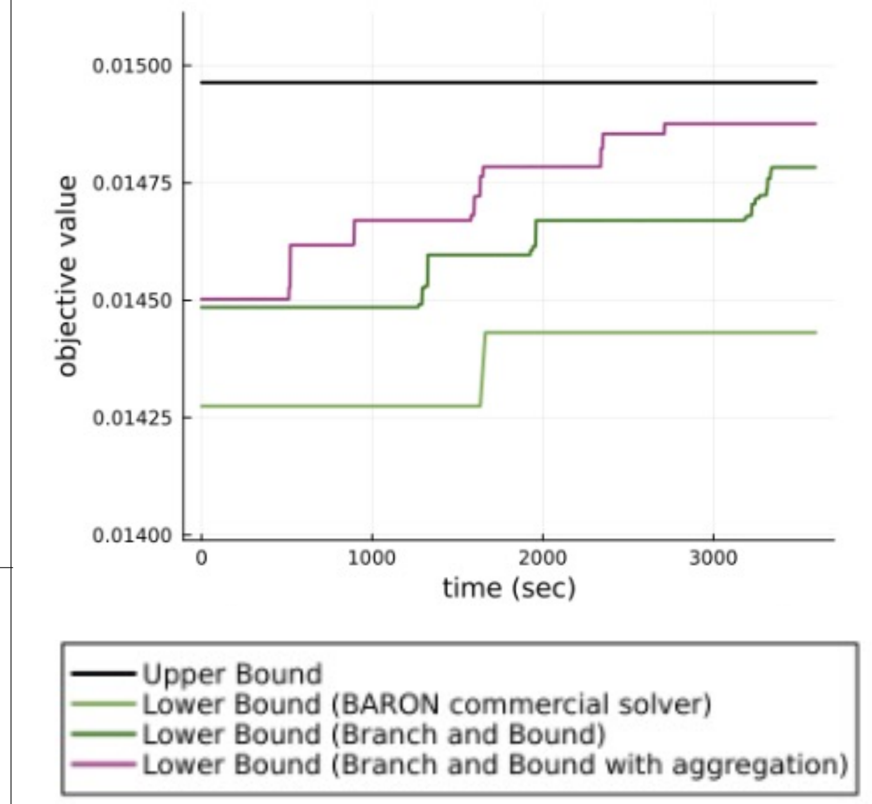
$$\begin{array}{lll} \min\limits_{\delta, x, y} & \delta & \\ \text{s.t.} & x^\top Q_i y + a_i^\top x + b_i^\top y + c_i = 0 & \forall i \in [n] \\ & \ldots \text{ other linear constraints w.r.t. } \delta, x, y & \end{array}$$



(a) Mathematical Models (b) As-built Structures

7 Aggregations can improve branch and bound convergence!

(a) Convergence of Example Instance



(b) Average Relative Improvements

	12story ($n_1 = 14$, $n_2 = 24$)	16story ($n_1 = 19$, $n_2 = 48$)
Root Node against Branch and Bound w/o aggregation		
	3.08%	6.65%
Final Gap against Branch and Bound w/o aggregation		
	8.33%	2.60%
Final Gap against BARON commercial solver		
	82.45%	51.18%

Remark: There is a trade-off between making aggregations and making the convex hull tighter, so we limit the number of aggregations to add.

Refenreces

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3. Santanu S Dey, Asteroide Santana, and Yang Wang. New SOCP relaxation and branching rule for bipartite bilinear programs. Optimization and Engineering, 20:307–336, 2019.