UF Herbert Wertheim College of Engineering
Department of Industrial & Systems Engineering
UNIVERSITY of FLORIDA

Symmetrically Fair Allocations of Indivisible Goods

Connor Johnston Aleksandr M. Kazachkov

THEORY

Problem: We wish to divide a set of m discrete goods amongst n agents. However, **we do not know which bundle of goods each agent will receive**.

What provable guarantees exist for the envy of the agents?

Example: A local nonprofit preallocates food donations into boxes that are picked at random by clients; ideally, the preallocation scheme would have a guarantee about the relative satisfaction with their bundles.

Definition: An allocation $(A_1, \ldots, A_n)$ is symmetrically envy free up to one good, or **symEF1**, when every agent $i \in [n]$ weakly prefers any bundle $k \in [n]$ over any other after removing their favorite item from that bundle. Concretely, for every $l \in [n]$ it holds that:

$$v_i(A_k) \geq v_i(A_l) - \bar{v}_i(A_l).$$

Goal: Derive sufficient condition for symEF1 existence.

Intuition: If all agents are identical, then a "round-robin allocation" provides a symEF1 allocation.

General case: Align agents round robin allocations.

First, construct an auxiliary structure for each agent i: a set $\mathcal{T}^i$ of "**indexed n-tuples**" $(\mathcal{T}_1^i, \ldots, \mathcal{T}_{m/n}^i)$ obtained by

1. Sort the items according to agent i's preferences.
2. Let $\mathcal{T}_1^i$ denote agent i's favorite n items.
3. Let $\mathcal{T}_2^i$ be the next-best n items according to agent i.
   …
4. Let $\mathcal{T}_{m/n}^i$ be the last n items according to agent i.

Definition: Allocation $\mathcal{A} = \{A_1, \ldots, A_n\}$ **separates** $\mathcal{T}^i$ if each bundle A_k contains an item from each of the indexed n-tuples T_1^i through $T_{m/n}^i$.

LEMMA

If an allocation A separates $\mathcal{T}^i$ for all $i \in [n]$, then A is symEF1 allocation.

Next, we construct a graph based on the tuples.

Definition: Let $\mathcal{T} = (\mathcal{T}^1, \ldots, \mathcal{T}^n)$, the **item graph** $G(\mathcal{T})$ has vertex set $[m]$ (one vertex per item) and edge set:
$\{\{j_l, j_k\} \subseteq [m] \times [m] : j_l \neq j_k \text{ and } \{j_l, j_k\} \subseteq \mathcal{T}^i\}$ for some $i \in [n]$.

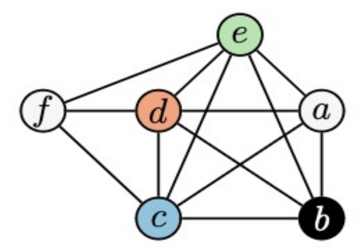
Main results:

THEOREM If $G(\mathcal{T})$ is n-colorable, then there exists a symEF1 allocation

COROLLARY For $n = 2$ agents, there always exists a symEF1 allocation

Separating tuples is sufficient, not necessary:

	Item a	Item b	Item c	Item d	Item e	Item f
Agent 1	1	2	3	4	5	6
Agent 2	1	2	4	3	5	6
Agent 3	1	2	4	5	3	6



The item item values highlighted for each agent represent the items in the set $\mathcal{T}_2^i$ for each agent i.

$$\mathcal{T}^1 = (\mathcal{T}_1^1, \mathcal{T}_2^1), \mathcal{T}_1^1 = \{f, e, d\}, \mathcal{T}_2^1 = \{c, b, a\}$$
$$\mathcal{T}^2 = (\mathcal{T}_1^2, \mathcal{T}_2^2), \mathcal{T}_1^2 = \{f, e, c\}, \mathcal{T}_2^2 = \{d, b, a\}$$
$$\mathcal{T}^3 = (\mathcal{T}_1^3, \mathcal{T}_2^3), \mathcal{T}_1^3 = \{f, d, c\}, \mathcal{T}_2^3 = \{e, b, a\}$$

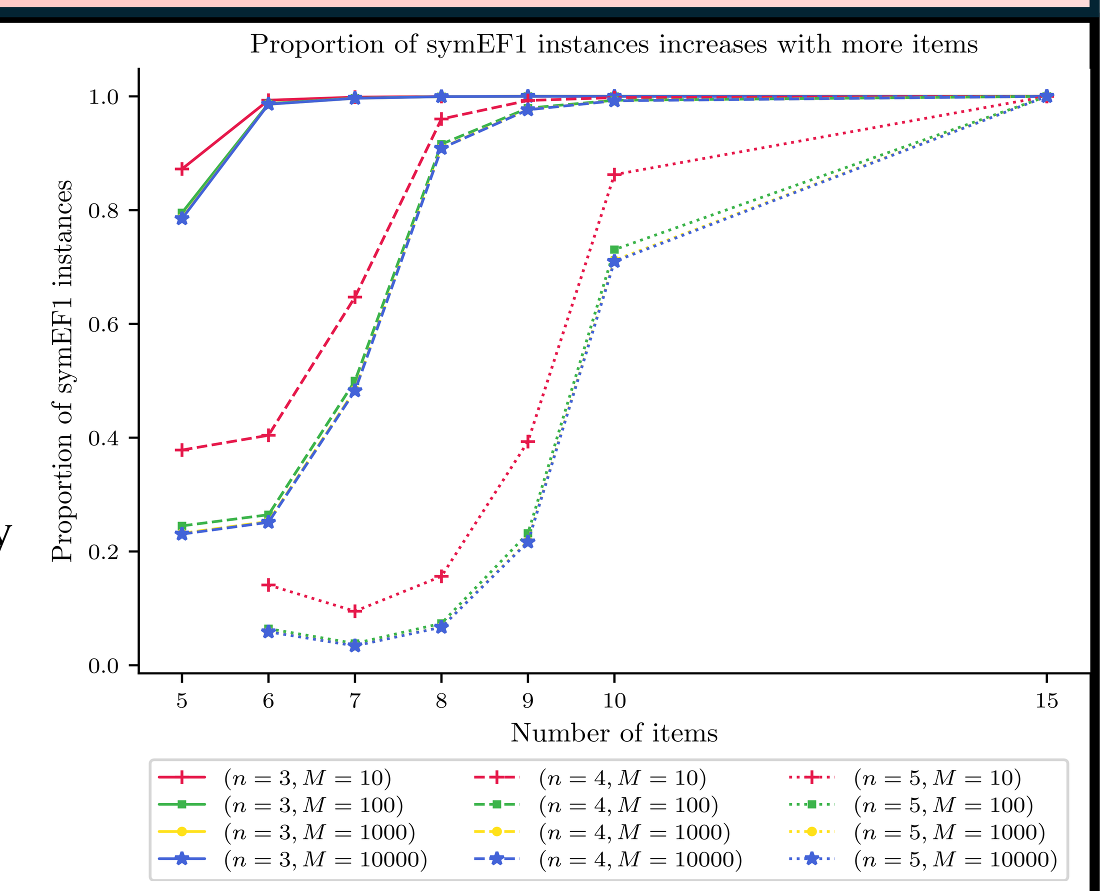
Let $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3)$, the item graph $G(\mathcal{T})$ is also shown.

We note that this example shows **our condition is not necessary** as $G(\mathcal{T})$ is not 3-colorable but a symEF1 allocation does exist.

COMPUTATION

Example: The valuation table below shows an instance in which a **symEF1 allocation does not exist.**

	Item a	Item b	Item c	Item d
Agent 1	1	1	1	0
Agent 2	1	1	0	1
Agent 3	1	0	1	1



Goal: Perform a simulation study to approximate the density of symEF1 allocations while varying both m and the valuations for the set of agents.

Verifying symEF1 allocations:

- The graph coloring condition cannot be used because the condition is sufficient but not necessary. Additionally, graph coloring is computationally expensive. To address this problem, we develop both an integer program as well as a heuristic.

Integer program:

- x_{kj}: binary variable representing that bundle $k \in [n]$ contains item $j \in [m]$.
- y_{ijl}: binary variable denoting that item $j \in [m]$ is removed by agent $i \in [n]$ from bundle $l \in [n]$.

$\sum_{k \in [n]} x_{kj} = 1$	for all $j \in [m]$	(one bundle per item)
$\sum_{j \in [m]} y_{ij\ell} \leq 1$	for all $i, \ell \in [n]$	(agent i can remove one item from bundle ℓ)
$y_{ij\ell} \leq x_{\ell j}$	for all $i, \ell \in [n]$, $j \in [m]$	(can only remove if item is present)
$\sum_{j \in [m]} v_{ij} x_{kj} \geq \sum_{j \in [m]} v_{ij}(x_{\ell j} - y_{ij\ell})$	for all $i, k, \ell \in [n]$, $k \neq \ell$	(agent i is EF1 with bundle k)

$$x \in \{0,1\}^{n \times m}, y \in \{0,1\}^{n \times m \times n}.$$

Heuristic: As integer programs can be computationally expensive, we develop a heuristic for finding feasible solutions, which iteratively extends a partial symEF1 allocation. At the end, either all items are assigned to bundles, or the heuristic returns "symEF1 allocation not found".

We consider three simple cases for extending a partial allocation with an item j not currently in a bundle:

- **Case 1:** add item j to an existing bundle;
- **Case 2:** add item j to a bundle after moving one item to another bundle;
- **Case 3:** add item j to a bundle after swapping items between two bundles.

Even for $n = 2$ agents, the heuristic may fail.

n	m	% symEF1	% Case 1	% Case 2	% Case 3	% IP	Time (s)
3	5	78.50	83.64	16.36	0.00	36.08	174.72
3	6	98.62	81.00	18.94	0.06	25.25	50.22
3	7	99.66	78.13	21.35	0.52	19.53	47.89
				⋮			
4	6	25.13	82.83	17.17	0.00	86.20	3,688.49
4	7	48.25	78.93	20.99	0.08	85.50	3,901.20
4	8	90.86	75.80	23.93	0.27	80.77	890.59
4	9	97.64	71.97	26.70	1.34	76.36	727.15
				⋮			
5	8	6.68	80.17	19.83	0.00	99.42	2,083.35
5	9	21.67	76.85	22.99	0.15	99.28	952.77
5	10	70.99	74.32	25.16	0.53	99.05	568.76
5	15	100.00	59.03	32.53	8.44	96.69	416.10
Average			77.26	21.22	1.52	97.89	

FUTURE WORK

Conjecture 1: *If there exists a symEF1 allocation for n agents and $m > n$ items, then there are at least two distinct symEF1 allocations.*

In our simulation experiments, the proportion of symEF1 instances increases rapidly with the number of items.

Conjecture 2: *For any fixed number of agents n, as the number of items $m \to \infty$, the probability that this instance has a symEF1 allocation goes to 1.*

While we have proved a sufficient condition for the existence of a symEF1 allocation, a symEFk allocation for $k > 1$ has eluded us so far.

Conjecture 3: *A symEF$(n-1)$ allocation always exists for any number of agents n and any number of items m.*

CONCLUSIONS

We introduce the fairness concept of symmetrically fair allocations. A symEF1 allocation is appealing in many settings such as food aid and of theoretical interest due to the sheer strength of the requirement.

While symEF1 allocations are not always guaranteed to exist for more than two agents, it is surprising how often it does exist. Additionally, it is surprising it always exists for two agents.

Furthermore, the concept of symmetrical allocations has been useful for group fairness research, and future research would benefit from better connections to this and consensus splitting results.