May 24, 2023

Integer points in polytopes are hard to find

Mixed Integer Programming
@USC

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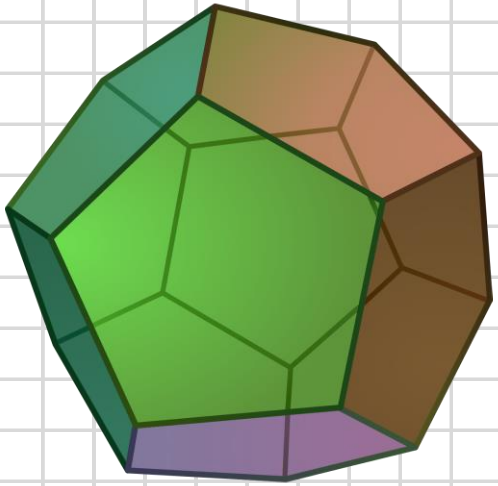
Joint work with Danny Nguyen



Link to my website.

Plan of the talk:

1) Previous work
2) Short story
3) Our results
4) Applications

Integer Programming in Fixed Dimension

Theorem (Lenstra, 1983) In $\mathbb{R}^d$, dimension d fixed, IP $\in$ P:

$$\text{(IP)} \qquad \exists \mathbf{x} \in \mathbb{Z}^d \, : \, A\mathbf{x} \leq \bar{b}.$$

Theorem (Barvinok, 1993) In $\mathbb{R}^d$, dimension d fixed, #IP $\in$ FP:

$$\text{(\#IP)} \qquad \#\{\mathbf{x} \, : \, A\mathbf{x} \leq \bar{b}\}.$$

Note: The system can be *long* here (i.e. has unbounded size)

Proof ideas: 1) Geometry of numbers (flatness theorem), lattice reduction (LLL).

2) Brion–Verge generating function approach, cone subdivisions, combinatorial tools.

Parametric Integer Programming

Theorem (Kannan, 1990) For all dimensions d, k fixed, PIP $\in$ P:

$$(\mathrm{PIP}) \qquad \forall \mathbf{y} \in Q \cap \mathbb{Z}^k \ \ \exists \mathbf{x} \in \mathbb{Z}^d \ : \ A\mathbf{x} + B\mathbf{y} \le \bar{b}.$$

Theorem (Barvinok–Woods, 2003) For all dimensions d, k fixed, #PIP $\in$ FP:

$$(\#\mathrm{PIP}) \qquad \#\left\{\mathbf{y} \in Q \cap \mathbb{Z}^k \ \ \exists \mathbf{x} \in \mathbb{Z}^d \ : \ A\mathbf{x} + B\mathbf{y} \le \bar{b}\right\}.$$

Let $P \subset \mathbb{R}^d$ be a convex polytope given by $A\mathbf{x} \le \bar{b}$. Say, $d = 3$.

Can one compute $\#\mathrm{E}(P)$ – the *number of integer points* in P? (Yes!)

Translation: These are $\mathrm{E}(Q) \subseteq_? \mathrm{E}(P)\!\downarrow$ and $\#\left[\mathrm{E}(Q) \cap \mathrm{E}(P)\!\downarrow\right]$.

Generalized Integer Programming

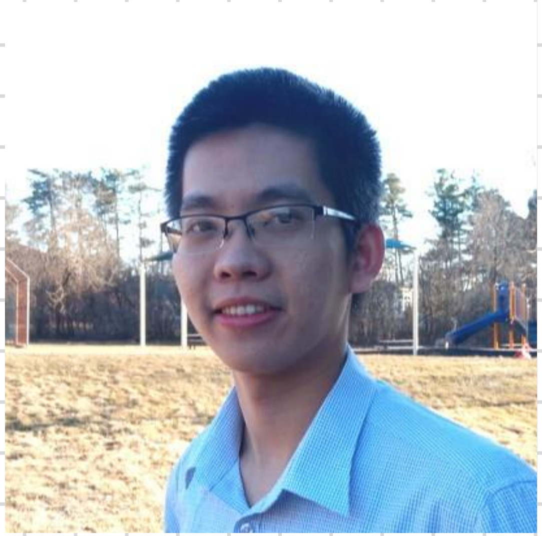
Open Problem (Kannan, 1990) Is GIP $\in$ P for all dimensions d, k, ℓ fixed?

$$\text{(GIP)} \qquad \exists \mathbf{z} \in R \cap \mathbb{Z}^{\ell} \;\; \forall \mathbf{y} \in Q \cap \mathbb{Z}^{k} \;\; \exists \mathbf{x} \in \mathbb{Z}^{d} \; : \; A\mathbf{x} + B\mathbf{y} + C\mathbf{z} \le \bar{b}.$$

Conjecture (Woods, 2003): This problem is in P.

A story:

1) Barvinok complained he cannot solve GIP
2) He complained again, and again
3) I suggested in might not be in P
4) He begged "take me out of this misery!"
5) I laughed and ignored him
6) He asked again, and again
7) Danny and I made it happen



First attempt:

STOC 2017 Accepted Papers

- Short Presburger arithmetic is in P
  Danny Nguyen, Igor Pak

RESEARCH-ARTICLE

Complexity of short Presburger arithmetic

Authors: Danny Nguyen, Igor Pak Authors Info & Claims

Theorem (Nguyen–P., STOC'17) KPT implies that GIP $\in$ P.

KPT = *Kannan's Partition Theorem* (1990) is the Main Lemma in the proof of Kannan's PIP Theorem.

Second attempt:

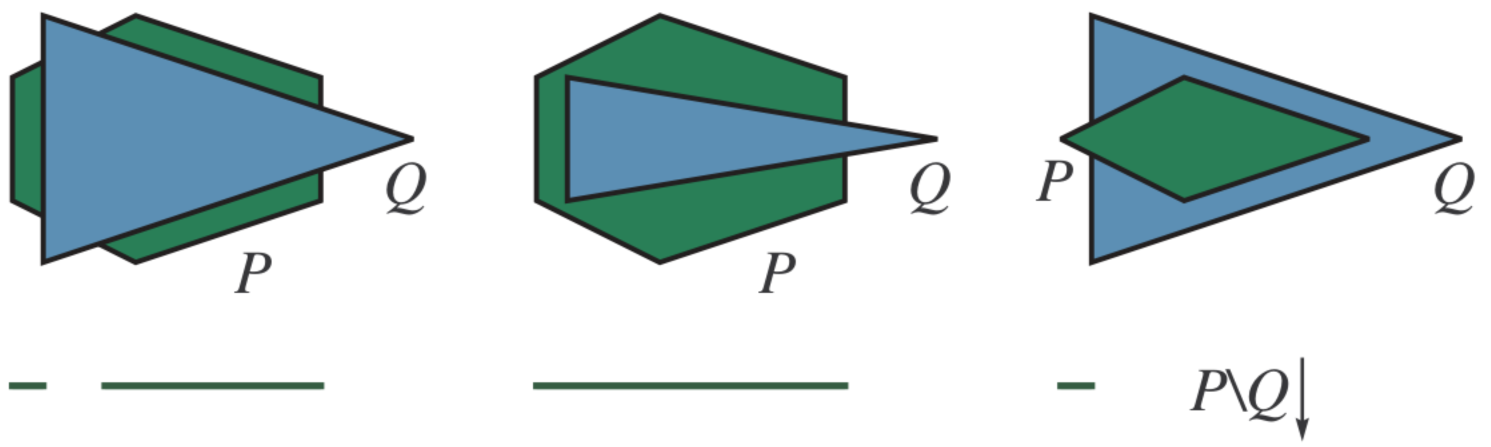
Theorem (Nguyen–P., CCC'17)

For dimensions $d \geq 3$, $k, \ell \geq 1$ fixed, Long–GIP is NP-complete.

The corresponding counting version #Long–GIP is #P-complete.

Theorem (Nguyen–P., CCC'17)

For $P, Q \in \mathbb{R}^3$, computing $\#\left[\mathrm{E}(P \smallsetminus Q)\downarrow_x\right]$ is #P-complete.



Third Attempt:

Theorem (Nguyen–P., FOCS'17)

Problem GIP is NP–complete.

Problem #GIP is #P–complete.

Notes: This is stronger than our CCC theorem.

With STOC theorem we have: KPT $\Rightarrow$ $\mathsf{P} = \mathsf{NP}$.

Theorem (Nguyen–P., FOCS'17)

KPT theorem is false.

Note: Kannan's PIP and Barvinok–Woods #PIP theorems remain true, see [Eisenbrand'03] and [Eisenbrand–Shmonin'08].

First application: bilevel optimization

Theorem 1.6. *Given a rational interval $J \subset \mathbb{R}$, a rational polytope $W \subset \mathbb{R}^5$ and a quadratic rational polynomial $h : \mathbb{R}^6 \to \mathbb{R}$, computing:*

$$\max_{z \in J \cap \mathbb{Z}} \quad \min_{\mathbf{w} \in W \cap \mathbb{Z}^5} \quad h(z, \mathbf{w}) \tag{1.1}$$

is NP*-hard. This holds even when W has at most 18 facets.*

Polynomial objective function

$$\min\{f^d(x) \,:\, x \in P \cap \mathbb{Z}^n\}$$

f^d is a polynomial of degree at most d

	$n = 1$	$n = 2$	$n = 58$	n fixed	n general
$d = 1$	P	P	P	P[a]	NPH[b]
$d = 2$	P	?	?	?	NPH
$d = 3$	P	?	?	?	NPH
$d = 4$	P	NPH[c]	Und[d]	Und	Und

Integer Quadratic Programming in the Plane

Alberto Del Pia Robert Weismantel

June 2, 2014

Second application: Pareto optima

Definition: [*Pareto minimum*]

Given polytope $Q \subset \mathbb{R}^n$ and functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ restricted to $Q \cap \mathbb{Z}^n$.

For $\mathbf{x} \in Q \cap \mathbb{Z}^n$, vector $\mathbf{y} = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ is called a *Pareto minimum* if:

- there is no other point $\widetilde{\mathbf{x}} \in Q \cap \mathbb{Z}^n$ and $\widetilde{\mathbf{y}} = (f_1(\widetilde{\mathbf{x}}), \ldots, f_k(\widetilde{\mathbf{x}}))$, such that $\widetilde{\mathbf{y}} \leq \mathbf{y}$ coordinate-wise and $\widetilde{\mathbf{y}} \neq \mathbf{y}$.

The goal: For the *objective function* $g : \mathbb{R}^k \to \mathbb{R}$,

minimize $g(\mathbf{y})$ over all Pareto minima $\mathbf{y}$ of $(f_1, \ldots, f_k)$ on Q.

Theorem 1.7. *Given a rational polytope $Q \subset \mathbb{R}^6$, two rational linear functions $f_1, f_2 : \mathbb{R}^6 \to \mathbb{R}$, a rational quadratic polynomial $f_3 : \mathbb{R}^6 \to \mathbb{R}$, and rational linear objective function $g : \mathbb{R}^3 \to \mathbb{R}$, computing the minimum of g over the Pareto minima of (f_1, f_2, f_3) on Q is* NP-*hard. Moreover, the corresponding* 1/2-*approximation problem is also* NP-*hard. This holds even when Q has at most 38 facets.*

Thank you!