Markov Chain-based Policies for Multi-stage Stochastic Integer Linear Programming with an Application to Disaster Relief Logistics

Margarita Castro

Joint work with:
Merve Bodur and Yongjia Song

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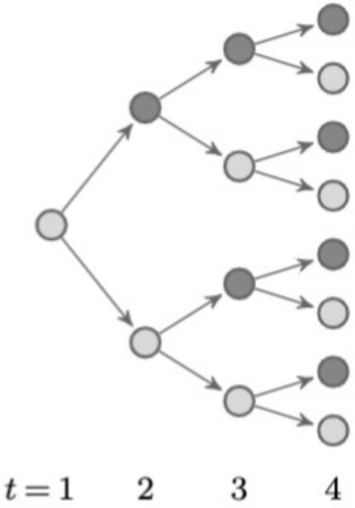
Multi-stage Stochastic Integer Programming

Scenario tree representation



$$\min \sum_{n \in N} p_n \cdot f_n\,(x_n, z_n, y_n)$$

$$s.t.\ (x_n, z_n, y_n) \in X_n(x_{a(n)}, z_{a(n)})$$

$y_n \in \mathbb{R}^m$ } Cont. local variables

$x_n \in \mathbb{R}^r$, $z_n \in \mathbb{Z}^l$ } Cont. and integer state variables

Assumptions:

- Linear constraints and objective
- Stochasticity given by a Markov Chain

Extremely challenging problems!!

How to Solve these Problems?

Exact techniques

- SDDiP (Zou, Ahmed & Sun, 2019)
- SDDP for MINLP (Zhang & Sun, 2022)

- Convexify the cost-to-go functions
- Some limitations on the implementation side

Our work:
Build an approximation with convex cost-to-go functions

Approximations

- Linear decision rules (LDR) (Kuhn et al., 2011)
- Two-stage LDR (Bodur & Luedtke, 2018)

- Transform into 1- or 2-stage stochastic problems
- Good approximations in practice
- Only for the continuous variables

Contributions

Main idea:

Create a partial extended formulation with only integer variables in the first stage

Aggregation framework:

- Impose additional structure to the integer variables based on the stochastic process (e.g., Markov Chain)

Methodology:

- Branch-and-cut algorithm integrated with SDDP.
  - Exact and approximated method.
- MC-based two-stage linear decision rules.
  - Approximated method.

Application:

- Hurricane disaster relief logistics planning.

Aggregation Framework

Claim:

The problem would be easier to solve if we only have 1st-stage integer variables.

Why?

- Piece-wise convex cost-to-go functions.
- Amenable for decomposition algorithms (e.g., SDDP)

Partially Extended Reformulation

$$\min \sum_{n \in N} p_n \cdot f_n\,(x_n, y_n, z)$$

$$s.t.\ (x_n, y_n) \in X_n(x_{a(n)}, z)$$

$y_n \in \mathbb{R}^m$, $x_n \in \mathbb{R}^r$ — Cont. state and local variables

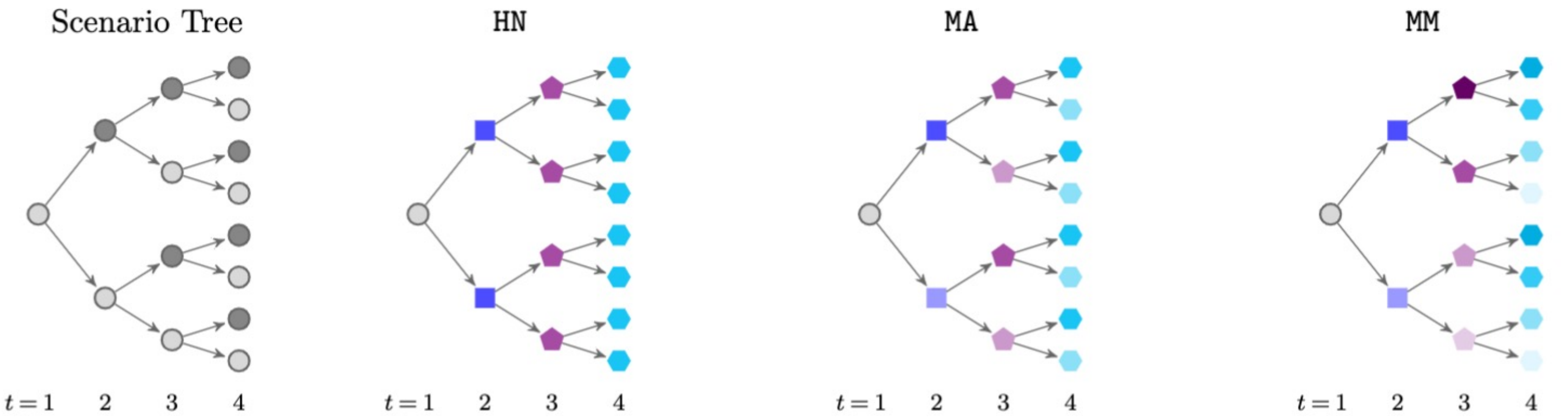
$z \in \mathbb{Z}^{l \times |N|}$ — Integer first–stage variables

Issue:

Too many first-stage variables!!

Aggregation Framework

Our Solution: Aggregate integer variables based on the underlying stochastic process (e.g., Markov Chain)



Here-and-now (current stage)

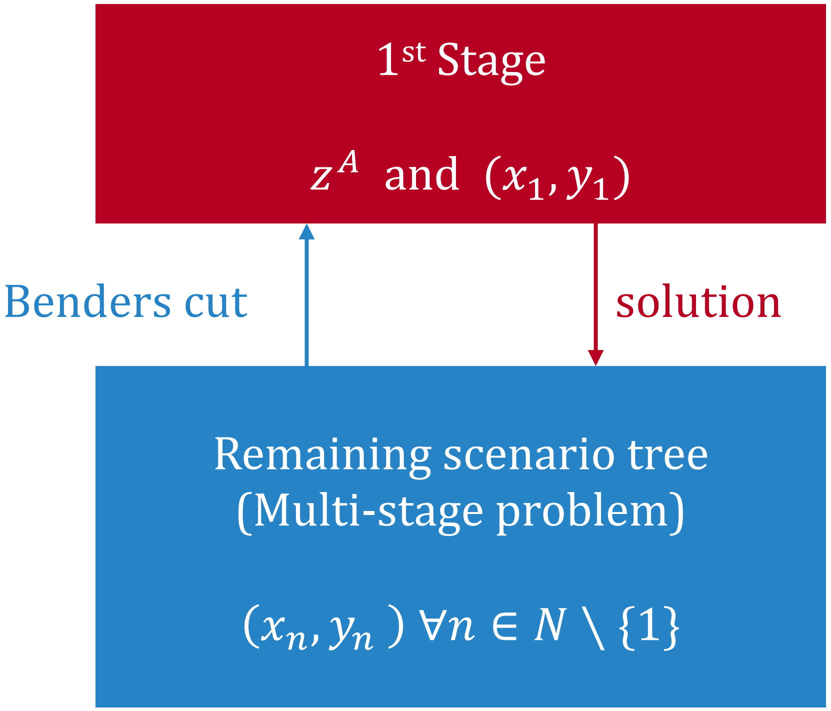
Markov-based (current MC state)

Previous and current MC state

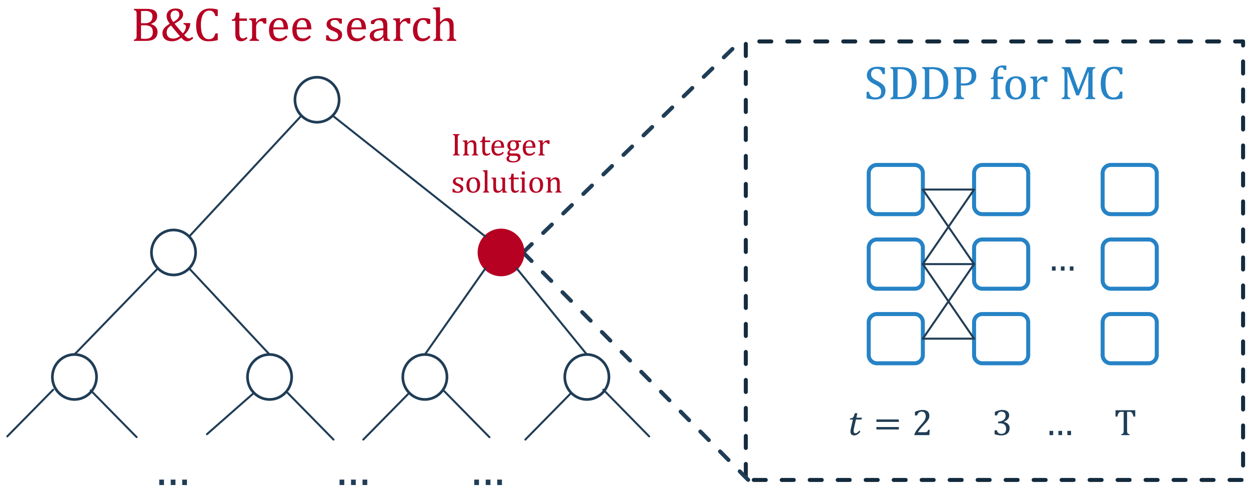
Methodology

Branch-and-Cut + SDDP

Decomposition for
the aggregated model



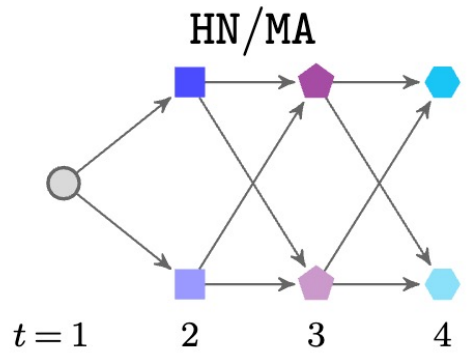
Branch-and-Cut (B&C)
+
Stochastic Dual Dynamic Programming (SDDP)

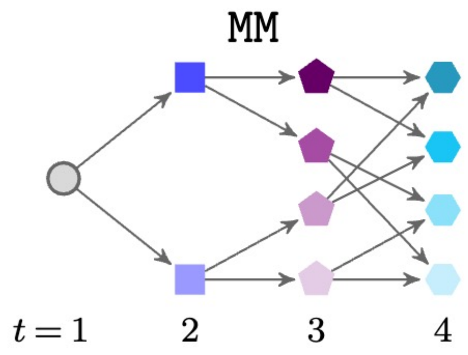


SDDP Algorithm

SDDP sub-problems

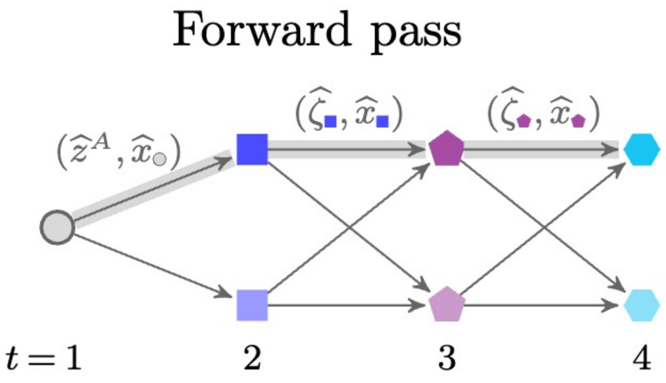
- One per MC-state
- Also affected by the aggregation

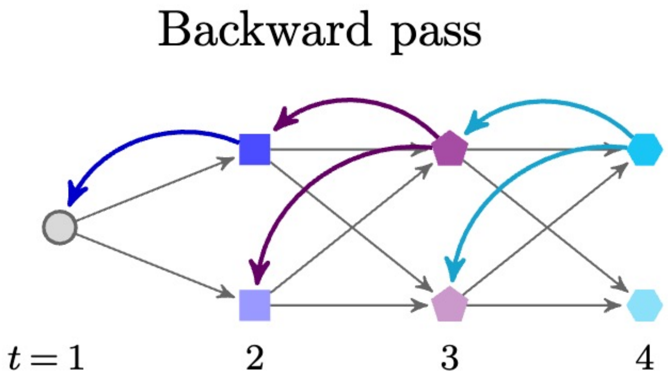




Forward and backwards pass

Computationally expensive!!





Two-Stage Linear Decision Rules (2S-LDR)

Goal:

Approximate a multi-stage problem with a two-stage problem using a linear transformation of the state variables
(Bodur & Luedtke, 2018)

- Random variable realizations:

$$\xi_n^t = \left(\xi_{1,n}, \dots, \xi_{t,n}\right) \quad \forall n \in N_t$$

- 2S-LDR:

$$x_n(\xi_n^t) = {\xi_n^t}^\top \mu \quad \forall n \in N_t$$

New 1[st] stage variables
for linear transformation

Three 2S-LDR alternatives

- Stage-history LDR:

$$x_n(\xi_n^t) = \mu_t \xi_n^t$$

- Stage-based LDR:

$$x_n(\xi_n^t) = \mu_t \xi_{t,n}$$

- Novel MC-based decision rule:

$$x_n(\xi_n^t) = \mu_{t,m(n)} \xi_{t,n}$$

Two-Stage Linear Decision Rules (2S-LDR)

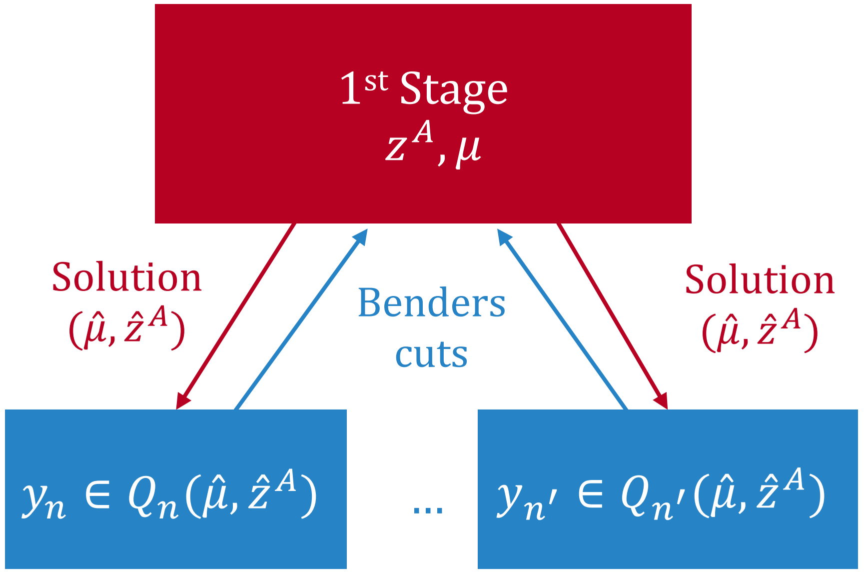
Aggregated framework and 2S-LDR

$$\min c^\top z^A + \sum_{n \in N} p_n \boxed{Q_n(\mu, z^A)}$$

← Cost-to-go function for each node

$$\begin{aligned} s.t. \quad & (\mu, z^A) \in X' & \\ & z_t^A \in \mathbb{Z}^{l \cdot q_t} & \forall t \in [T] \\ & \mu_t \in \mathbb{R}^{k \cdot l^t} & \forall t \in [T] \end{aligned}$$

Decomposition scheme



One sub-problem per node in the scenario tree

Hurricane Disaster Relief Planning

WITH CONTINGENCY MODALITY

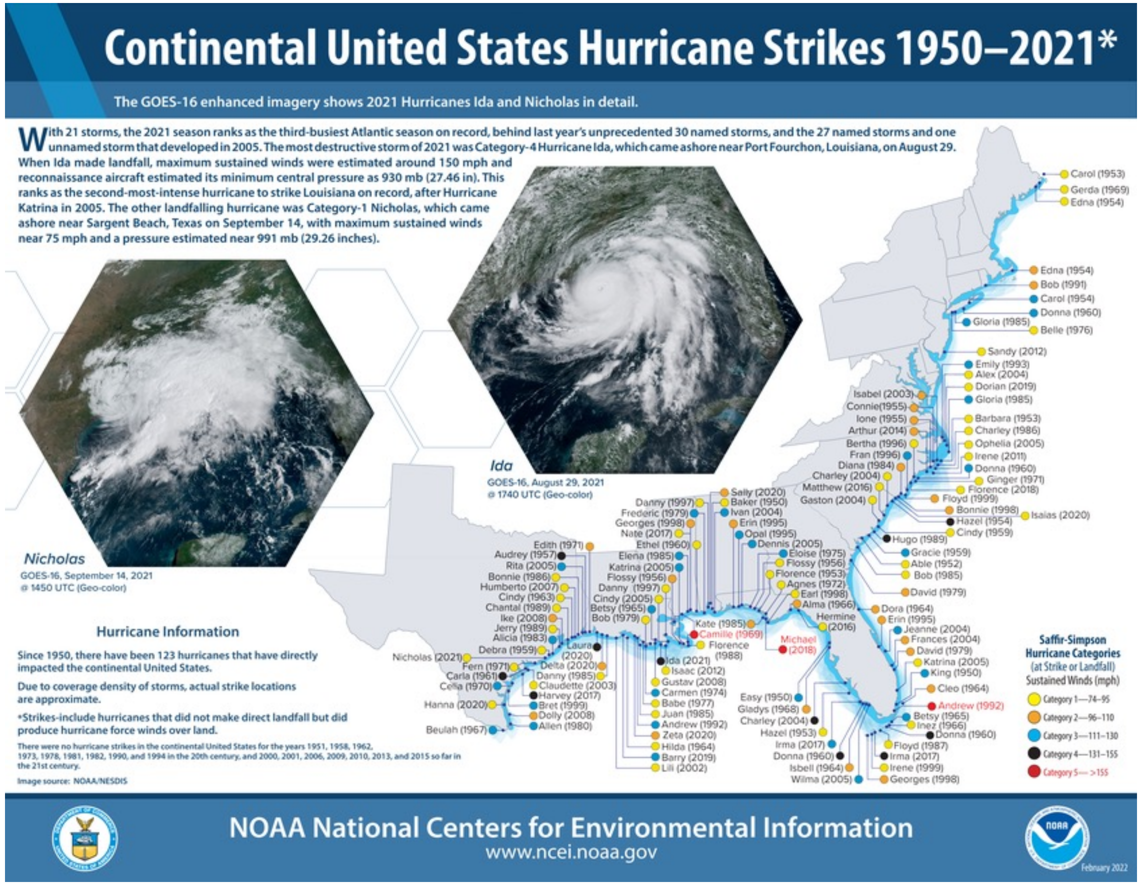
Hurricane Disaster Relief Planning

Description

- Produce and distribute resources from distribution centers (DCs) to shelters
- Multiple stage:
  - Start when the hurricane originates at sea and ends at landing
  - Update information of the hurricane in each stage

Objective: minimize cost

- Unsatisfied demand
- Transportation, production, and inventory



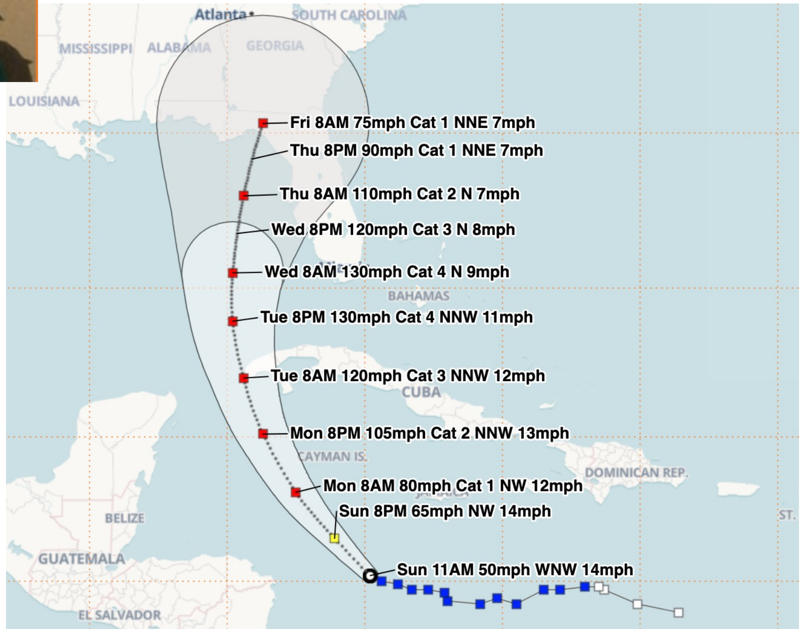
Hurricane Disaster Relief Planning

Contingency modalities

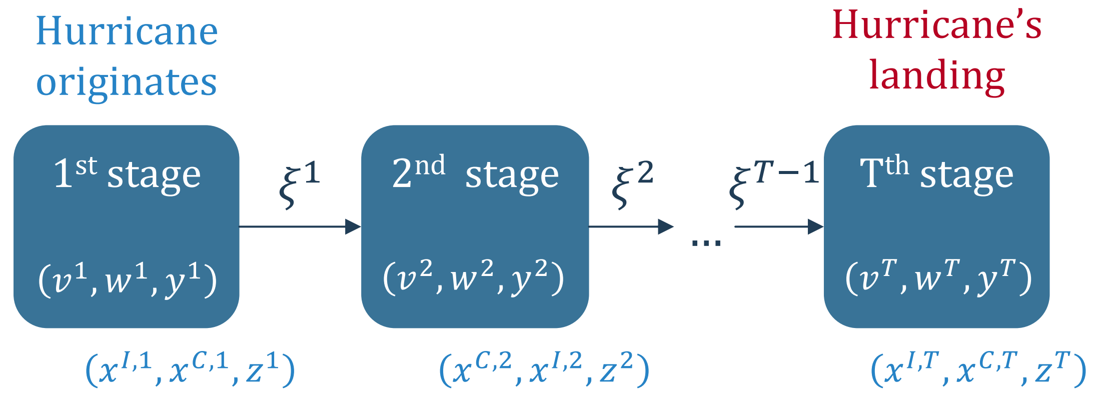
- Increase capacity of DCs:
  - Choose only one modality
  - Ones active, it stays active

Uncertainty

- Stochastic demand depends on hurricane intensity and location
- Evolution of the hurricane is given by a Markov chain (MC)



Multi-stage Stochastic Model



Local variables:

- Production (v), distribution (y), unsatisfied demand (w)

State variables:

- Contingency modality activation (z)
- Inventory (x^I) and capacity (x^C)

Scenario Tree Formulation

$$\min \sum_{n \in N} p_n \Big(\underbrace{F(x_n^I, y_n, w_n, v_n)}_{\text{Operational cost}} + \underbrace{\sum_{l \in L} z_n c_l}_{\text{Modality cost}} \Big)$$

$$s.t. \quad \sum_{l \in L} z_n \leq 1 \qquad \forall n \in N \qquad \text{Choose one modality}$$

$$z_{a(n)l} \leq z_{nl} \qquad \forall n \in N, l \in L \qquad \text{Modality remains active}$$

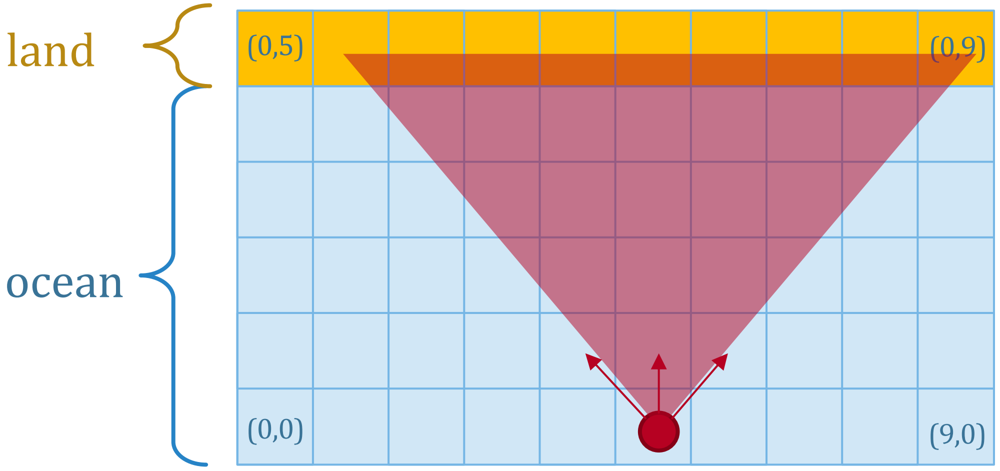
$$z_{nl} \in \{0,1\} \qquad \forall n \in N, l \in L$$

$$(x_n^I, x_n^C, z_n) \in \underbrace{X_n(x_{a(n)}^I, x_{a(n)}^C, z_{a(n)})}_{\text{Local variables and operational constraints}} \qquad \forall n \in N$$

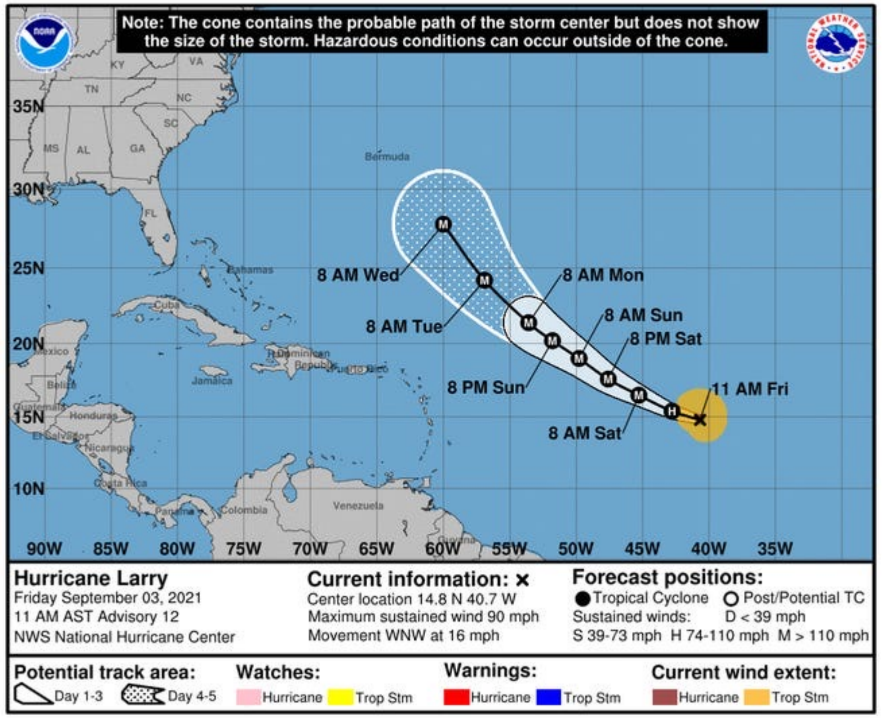
Stochasticity: Markov Chain

Markov Chain (MC) for the hurricane

- Region represented by a grid
- States: intensity + location
- Cone-shape movement until landing
- MC for intensity (Pacheco & Batta, 2016)



Initial state
Location: (7,0)
Intensity: 4



MC-based Tranformations and LDR

MC-based Transformations for Modalities

- HN: stage-based

$$z^{A}_{tl}$$

- MA: MC-based

$$z^{A}_{tm_t(n)l}$$

- MM: double MC-based

$$z^{A}_{tm_t(n)m_{t-1}(a(n))l}$$

- PM: MC + Intensity

$$z^{A}_{tm_t(n)m^{i}_{t-1}(a(n))l}$$

2S-LDR for Inventory

- Stage-based (LDR-T):

$$x^{I}_{nj}(d_n) = \sum_{i \in I} d_{ni} \mu_{t(n)ji}$$

- Stage + history (LDR-TH) :

$$x^{I}_{nj}(d_n) = \sum_{i \in I} \sum_{n' \in P(n)} d_{ni} \mu_{t(n')tji}$$

- Stage + MC state:

$$x^{I}_{nj}(d_n) = \sum_{i \in I} d_{ni} \mu_{tm_t(n)ji}$$

Empirical Results

Experimental Set-up

Experiments

- CPLEX 20.1 + callback
- Single thread
- Time limit: 6 hour

Techniques

- Extensive model
- B&C + SDDP
- 2S-LDR

Instances

- Small size: 4x5 grid and 5 stages
- Large size: 5x6 grid and 6 stages
- 6 level of intensity
- Initial capacity:
  - 20%, 25%, 30% of maximum demand
- Modality options:
  - Setting 1: 10%, 20%, 30% or 40%
  - Setting 2: 15%, 30%, 45% or 60%
- 10 instances per configuration

Value of MC-based Policies

		Average Objective Value					% Gap closed		
Modality	Cap.	HN	MA	PM	MM	FH	MA	PM	MM
Type-1	20%	104,162	102,572	92,924	92,924	82,193	7.5	51.9	51.9
	25%	73,879	73,442	66,650	66,238	62,972	3.2	69.4	73.1
	30%	48,970	48,951	47,425	47,314	47,117	0.3	89.5	96.7
Type-2	20%	104,135	102,625	92,612	92,612	81,182	6.7	50.9	50.9
	25%	73,919	73,487	66,573	65,958	63,654	3.3	75.6	81.2
	30%	48,970	48,968	47,540	47,349	47,253	0.0	80.3	97.0

0% HN — 100% Original

HN: Here-and-now transformation
MA: MC-based transformation considering current MC state
PM: MC-based transformation considering current MC state and previous state intensity
MM: MC-based transformation considering current and previous MC state
FH: Full-history multi-stage problem

Exact Methods – Extensive (Ex) vs. SDDP (S)

		Average Time (sec)								Gap (%)	
		HN		MA		PM		MM		PM	MM
Modality	Cap.	Ex	S	Ex	S	Ex	S	Ex	S	S	S
Type-1	20%	**260**	1,221	**712**	5,969	**961**	4,553	**1,853**	-	-	14.9
	25%	**132**	195	**361**	1,313	**487**	2,760	**626**	-	-	3.5
	30%	**73**	102	**109**	272	**222**	3,380	**367**	-	-	2.1
Type-2	20%	**253**	1,138	**919**	5,880	**2,047**	13,224	**2,951**	20,699	19.9	17.8
	25%	**167**	140	**393**	1,324	**651**	3,635	**817**	-	-	6.2
	30%	**82**	102	**119**	205	**271**	4,347	**556**	-	-	2.3
Average		**161**	483	**436**	2,494	**773**	5,316	**1,195**	20,699	19.9	7.8

Poor performance of SDDP due to large number of sub-problems:
69 for HN and 294 for MM

HN: Here-and-now transformation
MA: MC-based transformation considering current MC state
PM: MC-based transformation considering current MC state and previous state intensity
MM: MC-based transformation considering current and previous MC state
FH: Full-history multi-stage problem

Exact Methods – Extensive (Ex) vs. SDDP (S)

		# Optimal				# Feasible				Opt. Gaps (%)			
Modality	Cap.	HN	MA	PM	MM	HN	MA	PM	MM	HN	MA	PM	MM
Type-1	20%	2	0	0	0	8	10	7	1	53.4	51.8	56.1	82.0
	25%	5	2	0	0	5	8	6	0	35.7	26.2	27.9	-
	30%	10	8	0	0	0	1	4	0	-	15.4	4.3	-
Type-2	20%	3	0	0	0	7	10	7	1	43.5	52.9	61.4	86.0
	25%	6	3	0	0	4	7	6	0	37.2	24.9	32.8	-
	30%	9	8	0	0	1	1	6	0	12.0	13.3	6.4	-
Total/Av.		35	21	0	0	25	37	36	2	36.4	30.7	31.5	84.0

Extensive model cannot solve larger instances.

HN: Here-and-now transformation
MA: MC-based transformation considering current MC state
PM: MC-based transformation considering current MC state and previous state intensity
MM: MC-based transformation considering current and previous MC state
FH: Full-history multi-stage problem

Approximated Methods

Table 3 Solution time and quality of 2SLDR and SDDP bounds. Results for PM over small-size instances.

		Average Time (sec)					Relative Difference (%)				
Modality	Cap.	LDR-TH	LDR-T	LDR-M	S-LB	Ex	LDR-TH	LDR-T	LDR-M	S-UB	S-LB
Type-1	20%	813.0	164.4	**157.9**	549.1	961.3	0.12	0.26	0.11	**0.01**	0.36
	25%	380.1	84.3	**77.7**	347.0	487.2	0.08	0.11	**0.03**	0.26	0.71
	30%	408.2	**79.0**	85.9	400.7	222.0	0.12	0.25	**0.00**	0.55	1.24
Type-2	20%	1406.0	**247.2**	265.2	940.5	2047.3	0.13	0.26	0.13	**0.00**	0.28
	25%	670.5	108.2	**98.1**	534.4	651.0	0.09	0.13	**0.04**	0.23	0.49
	30%	532.6	90.0	**89.5**	434.8	270.6	0.12	0.26	**0.01**	0.98	1.29
Average		701.7	**128.9**	129.1	534.4	773.2	0.11	0.21	**0.05**	0.34	0.73

LDR:
High-quality solutions and small computational times

S-LB & S-UB:
Lower and upper bounds based on SDDP algorithm. Expensive but effective.

Approximated Methods

Table 4 Solution time and quality of 2SLDR and SDDP bounds. Results for PM over large-size instances.

		Average Time (sec)				Opt. Gap (%)		
Modality	Cap.	LDR-T	LDR-M	S-LB	(opt)	LDR-T	LDR-M	S-UB
Type-1	20%	5,353	**3,732**	13,564	(9)	0.62	0.39	**0.24**
	25%	**4,235**	4,716	13,557	(6)	2.56	2.06	**1.91**
	30%	**1,555**	1,675	2,067	(10)	0.80	0.63	**0.27**
Type-2	20%	7,489	**4,855**	18,323	(5)	9.00	**7.33**	9.01
	25%	6,664	**5,560**	14,790	(4)	4.98	4.38	**4.28**
	30%	1,717	**1,640**	6,971	(10)	0.81	0.63	**0.30**
Av. (Total)		4,502	**3,696**	11,545	(44)	3.13	**2.57**	2.67

Summary

- Aggregation framework for MSILP with mixed-integer state variables.
  - Reformulation and aggregation of integer variables in the 1st stage.
- Several transformations based on the stochastic process (Markov chain).

- B&C framework integrated with the SDDP algorithm.
- MC-based 2LDR.

- Hurricane disaster relief planning applications.
- Extensive empirical results showing trade-offs.

Thank you!

Questions?

Paper available in OO and ArXiv