INTERIOR POINT METHODS ARE NOT (MUCH) WORSE THAN SIMPLEX

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MIP 2022, Rutgers
Interior point method vs Simplex
<table>
<thead>
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<th>Simplex</th>
<th>IPM</th>
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<td><strong>Dantzig, 1947</strong></td>
<td><strong>Karmarkar 1984</strong></td>
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<td>Fast in practice</td>
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<tr>
<td><strong>Exponential in worst case</strong></td>
<td>Polynomial in the input size</td>
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<td>Polynomial smoothed complexity</td>
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<td>Easy to warm start</td>
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<td>Numerically stable</td>
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Complexity of LP algorithms

- $n$ variables, $m$ equality constraints
- Total encoding $L$.
- Worst case bound for Simplex: $\binom{n}{m} \leq 2^n$
- Worst case bound for IPMs: $\text{poly}(n, L)$

...lots of recent improvements, see Yin Tat’s talk
Complexity of LP algorithms

- Some problems with polynomial encoding can be formulated as LPs with exponential entries
- Is there any function $f(n)$ and an IPM method with running time $\min(p\text{oly}(n,L), f(n))$?
- Strongly polynomial algorithm: $\text{poly}(n)$ arithmetic operations

\[
\begin{align*}
\min c^T x \\
A x &= b \\
x &\geq 0
\end{align*}
\]
Is there a strongly polynomial algorithm for Linear Programming?

Smale’s 9th question
Strongly polynomial algorithms for classes of Linear Programs

$$\min c^T x, \quad Ax = b \quad x \geq 0$$

- Combinatorial problems: two variable per inequality systems, network flows, discounted MDPs, ...
- Tardos ’86: $\text{poly}(n, \log \Delta_A)$ dependence only on $A$, but not on $b$ and $c$.
  $$\Delta_A = \max \{|\det(B)| : B \text{ submatrix of } A\}$$
- Layered-least-squares (LLS) Interior Point Method
  Vavasis & Ye ’96: $\text{poly}(n, \log \bar{\chi}_A)$ LP algorithm in the real model of computation
  $\bar{\chi}_A$: Dikin–Stuart–Todd condition number
Primal and dual LP

- $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^m, b \in \mathbb{R}^n$

\[
\begin{align*}
\min c^T x & \quad \max b^T y \\
Ax &= b & A^T y + s &= c \\
x &\geq 0 & s &\geq 0
\end{align*}
\]

- Complementary slackness: Primal and dual solutions $(x, s)$ are optimal if $x^T s = 0$:
  \[x_i = 0 \text{ or } s_i = 0 \text{ for each } i \in [n].\]

- Optimality gap:
  \[c^T x - b^T y = x^T s.\]
The central path

- For each $\mu > 0$, there exists a unique $z(\mu) = (x(\mu), y(\mu), s(\mu))$ such that $x(\mu)_i s(\mu)_i = \mu \quad \forall i \in [n]$

  the central path element for $\mu$.

- The central path is the algebraic curve \{\(z(\mu) : \mu > 0\}\}

- For $\mu \to 0$, the limit is an optimal solution $z^* = (x^*, y^*, s^*)$.

- The duality gap is $s(\mu)^T x(\mu) = n\mu$.

- Interior point algorithms: walk down along the central path with $\mu$ decreasing geometrically.
The Mizuno–Todd–Ye Predictor-Corrector Algorithm

- Start from point \( z_0 = (x_0, y_0, s_0) \) 'near' the central path at some \( \mu_0 > 0 \).
- Alternate between
  - **Predictor steps**: 'shoot down' the central path, decreasing \( \mu \) by a factor at least \( 1 - 1/O(n) \). May move slightly 'farther' from the central path.
  - **Corrector steps**: do not change parameter \( \mu \), but move back 'closer' to the central path.

Within \( O(\sqrt{n}) \) iterations, \( \mu \) decreases by a factor 2.
Layered Least Squares Interior Point Method
Vavasis-Ye ’96

- $O(n^{3.5} \log \bar{\chi}_A)$ iterations
- $\bar{\chi}_A$: Dikin–Stuart–Todd condition number
- Combinatorial structure of central path:
  $\leq \binom{n}{2}$ short and curved segments connected by long and straight parts

Short = $O(n^{1.5} \log \bar{\chi}_A)$

LLS IPM glides through the long parts in $O(n^{1.5} \log \bar{\chi}_A)$
Scaling invariant bounds

\[ \bar{\chi}_A^* = \inf \{ \bar{\chi}_{AD} : D \in \mathbb{R}^{n \times n} \text{ positive diagonal} \} \]

- Central path is invariant under diagonal rescalings, but \( \bar{\chi}_A \) and the Vavasis–Ye algorithm are not.
- \( \text{poly}(n, \log \bar{\chi}_A^*) \) algorithms
  - Predictor-Corrector Trust Region algorithm
    Lan, Monteiro & Tsuchiya ’09
    computing the step directions has weakly polynomial dependence on \( b \) and \( c \)
  - Scaling invariant LLS
    Dadush, Huiberts, Natura, V. ’20
    using combinatorics of circuit imbalances
- \( \text{poly}(n, \log \bar{\chi}_A^*) \) bound on the Sonnevend–Stoer–Zhao curvature of the central path
  Monteiro & Tsuchiya ’08
$2^n$ vs $\text{poly}(n, \log \bar{\chi}_A^*)$

- $\bar{\chi}_A^*$ can be still unbounded
- Is $\text{poly}(n, \log \bar{\chi}_A^*)$ a tight bound on what an IPM can achieve, or...
- Could even strongly polynomial IPMs exist?
Theorem (Allamigeon, Benchimol, Gaubert & Joswig ’18): No path following method can be strongly polynomial that stays in the wide neighbourhood of the standard log barrier central path.

Proof using tropical geometry: studies the tropical limit of a family of parametrized linear programs.

Allamigeon, Gaubert & Vandame ’22: extension to arbitrary self-concordant barrier functions
IPM analogue of Klee–Minty cube

Previous work: Deza, Terlaky & Zinchenko ’09
There exists a primal-dual path following method where the number of iterations is

$$O(n^{1.5} \log n) \min \left\{ O(2^n), \min \text{no. of iterations of any path following method} \right\}$$
Following a piecewise linear path in the wide neighbourhood

For any piecewise linear curve with $T$ pieces on $(0, \mu_0)$ in the $\theta$-wide neighbourhood, our IPM makes at most

$$O \left(n^{1.5T} \log \frac{n}{1 - \theta}\right)$$

iterations is the $\ell_2$-neighbourhood.

\[
\bar{\mu}(z) = \frac{x^T s}{n}
\]

\[
CP = \left\{ z = (x, y, s): \frac{x_s}{\bar{\mu}(z)} = 1 \right\}
\]

\[
N^2(\beta) = \left\{ z = (x, y, s): \left\| \frac{x_s}{\bar{\mu}(z)} - 1 \right\| \leq \beta \right\}
\]

\[
N^{-\infty}(\theta) = \left\{ z = (x, y, s): \frac{x_s}{\bar{\mu}(z)} \geq (1 - \theta) \mathbf{1} \right\}
\]
The Max Central Path

**GOAL:** Show the existence of a piecewise linear curve with \( \leq 2^n \) pieces in the wide neighbourhood

Max Central Path
\[
\bar{x}(\mu) = (\bar{x}_1(\mu), \bar{x}_2(\mu), \ldots, \bar{x}_n(\mu)),
\]
\[
\bar{s}(\mu) = (\bar{s}_1(\mu), \bar{s}_2(\mu), \ldots, \bar{s}_n(\mu))
\]

\[
\bar{x}_i = \max x_i \quad \bar{s}_i = \max s_i
\]
\[
Ax = b \quad A^\top y + s = c
\]
\[
x \geq 0 \quad s \geq 0
\]
\[
c^\top x \leq OPT + n\mu \quad b^\top y \geq OPT - n\mu
\]

**LEMMA:** \( \frac{\bar{x}(\mu)}{2n} \leq x(\mu) \leq \bar{x}(\mu) \) and \( \frac{\bar{s}(\mu)}{2n} \leq s(\mu) \leq \bar{s}(\mu) \)

\( \bar{x}(\mu) \) and \( \bar{s}(\mu) \) not primal & dual feasible, but averaging the corresponding solutions gives a PL curve in the wide neighbourhood
The Max Central Path

- The Max Central Path has $O(2^n)$ segments
- It is also bounded by the total length of $2n$ shadow vertex simplex paths
- Shadow Vertex Simplex: average case analysis Borgwardt ’87 smoothed complexity Spielman & Teng ’04

- The Max Central Path is also related to the tropical central path used in the lower bounds
Following a piecewise linear path in the wide neighbourhood

For any piecewise linear curve with $T$ pieces on $(0, \mu_0)$ in the $\theta$-wide neighbourhood, our IPM makes at most

$$O \left( n^{1.5} T \log \frac{n}{1 - \theta} \right)$$

iterations is the $\ell_2$-neighbourhood.

\[ \tilde{\mu}(z) = \frac{x^T s}{n} \]

\[ CP = \left\{ z = (x, y, s): \frac{x s}{\tilde{\mu}(z)} = 1 \right\} \]

\[ \mathcal{N}^2(\beta) = \left\{ z = (x, y, s): \left\| \frac{x s}{\tilde{\mu}(z)} - 1 \right\| \leq \beta \right\} \]

\[ \mathcal{N}^{-\infty}(\theta) = \left\{ z = (x, y, s): \frac{x s}{\tilde{\mu}(z)} \geq (1 - \theta)1 \right\} \]
Polarization of central path segments

If there is a linear segment in the wide neighbourhood of the central path segment $CP[\mu_1, \mu_0] = \{x(\mu): \mu_1 \leq \mu \leq \mu_0\}$, then this segment is polarized.

$\exists B \cup N = \{1, 2, \ldots, n\}$

$\gamma x_i(\mu_0) \leq x(\mu) \leq n x_i(\mu_0) \quad \forall i \in B$

$\frac{1}{\mu} x_i(\mu_0) \leq x(\mu) \leq \frac{1}{\gamma \mu_0} x_i(\mu_0) \quad \forall i \in N$

same for $s(\mu)$ with $B$ and $N$ swapped.
Subspace Layered Least Squares IPM

New LLS step direction that can traverse any polarized segment $CP[\mu_1, \mu_0]$ in $O(n^{1.5} \log n)$ iterations, no matter the length.

**STEP 1:** guess the polarizing partition $B \cup N$

$(\Delta x^a, \Delta s^a)$: Standard affine scaling step

$\tilde{B} := \left\{ i : \left| \frac{\Delta x^a_i}{x_i} \right| < \left| \frac{\Delta s^a_i}{s_i} \right| \right\}$, $\tilde{N} := [n] \setminus \tilde{B}$

**LEMMA (Roughly):** If we are still far from the end of the polarized segment, this reveals the polarizing partition:

$\tilde{B} = B$, $\tilde{N} = N$
Predictor-Corrector Trust Region algorithm
Lan, Monteiro & Tsuchiya ’09

Given \((B, N)\), select the primal direction \(\Delta x \in \ker(A)\) such that

- make the most progress in decreasing variables in \(N\), while
- barely change the variables in \(B\)

Analogously for \(\Delta s \in \text{im}(A^T)\)

**PROBLEM:** cannot compute in strongly polynomial time
Subspace Layered Least Squares IPM

Given \((B, N)\), select the primal direction \(\Delta x \in \ker(A)\) such that

- make the most progress in decreasing variables in \(N\), restricting \(\Delta x_N \in V\) for a subspace \(V \subseteq \text{Proj}_N(\ker(A))\)
- \(V\) is chosen such that any \(\Delta x_N \in V\) can be extended to \((\Delta x_B, \Delta x_N) \in \ker(A)\) with small \(\Delta x_B\)
- \(V\) obtained using an (approximate) singular value decomposition
Focus on subspaces $V \subseteq \text{Proj}_N(\ker(A))$ and $U \subseteq \text{Proj}_B(\text{im}(A^T))$ used in the algorithm.

In $O(\sqrt{n} \log n)$ iterations

- We reach the end of the polarized segment, or
- $\dim(V)$ and $\dim(U)$ increase

Can happen at most $n$ times
\begin{align*}
\min \left\{ \text{poly}(n, \log \bar{\chi}_A^*), \ 2^n \right\}
\end{align*}

**THEOREM** (Allamigeon, Dadush, Loho, Natura & V. ’22)

There exists a primal-dual path following method where the number of iterations is

\[ O(n^{1.5} \log n) \min \left\{ O(2^n), \min \text{ no. of iterations of any path following method} \right\} \]

The same holds for the Lan–Monteiro–Tsuchiya trust region algorithm.
Summary and open questions

- Have IPM beaten Simplex yet? **Not quite:**
  - Subexponential randomized simplex: Kalai ’92, Matoušek, Sharir & Welzl ’92
  - IPM has exponential lower bounds
- The number of iterations of our algorithm is universally the best for any log barrier IPM—up to a factor $O(n^{1.5} \log n)$

**QUESTIONS**

- Improving on $\bar{\lambda}_A^*$, can we find a tighter condition number that is always bounded as $O(2^n)$?
- Does our algorithm obtain new strongly polynomial algorithms for some problems?
- Find a tighter analysis of our algorithm
- Find such “universal” IPM for any self-concordant barrier
- Develop a strongly polynomial variant of the trust region step