Monoidal Strengthening

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MIP Workshop · May 24, 2022
What is Monoidal Strengthening?

- Introduced by Balas and Jeroslow in 1980
- It is a technique for strengthening intersection cuts by exploiting the integrality of non-basic integer variables
• We can deduce conditions over the feasible set of a MIP of the form

\[ \sum r^j x_j \in S \]

\[ x_j \geq 0 \]

where \( S \) is closed and \( 0 \notin S \)
• Consider the constraints $x_j \geq 0$ and the disjunction

\[ \sum_j \alpha_j x_j \geq 1 \]

\[ \lor \sum_j \beta_j x_j \geq 1 \]

• This is equivalent to $x_j \geq 0$ and

\[ \sum_j \left( \begin{array}{c} \alpha_j \\ \beta_j \end{array} \right) x_j \in S := \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \} \]
Example: Disjunctive cut

- Let $\alpha = (1, -2, 1)$ and $\beta = (-3, -\frac{1}{2}, 2)$
- Disjunction $\sum_j \alpha_j x_j \geq 1 \lor \sum_j \beta_j x_j \geq 1$
- Equivalently $\sum_j \left( \begin{array} \alpha_j \\ \beta_j \end{array} \right) x_j \in S := \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \}$
Example: Disjunctive cut

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- A feasible solution must describe a path into $S$
- For example, $x = (\frac{1}{2}, \frac{3}{5}, \frac{3}{2})$
A convex set $C$ is an $S$-free set if it does not contain any point of $S$ in its interior.

Thus, if $x$ is going to be feasible for \( \left( \sum r^j x_j \in S \right) \), it better be that $x_j \geq 0$ and $\sum r^j x_j \notin \text{int}(C)$.
Representations of $S$-free set and intersection cuts

- $\phi$ is a sublinear if it is
  - positively homogeneous: $\phi(rx) = \phi(r)x$ for all $x \geq 0$
  - subadditive: $\phi(\sum_j r^j) \leq \sum_j \phi(r^j)$

- As $C$ is convex with 0 in its interior, there is $\phi$ sublinear such that
  \[
  C = \{ x : \phi(x) \leq 1 \}
  \]
  \[
  (\text{int } C)^c = \{ x : \phi(x) \geq 1 \}
  \]

- If $x_j \geq 0$ and $\sum_j r^j x_j \notin \text{int}(C)$, then
  \[
  1 \leq \phi(\sum_j r^j x_j) \leq \sum_j \phi(r^j x_j) = \sum_j \phi(r^j) x_j
  \]

- Intersection cut:
  \[
  1 \leq \sum_j \phi(r^j) x_j
  \]
• $\phi(r) = \|r\|$ is sublinear and $C = \{r : \phi(r) \leq 1\}$
Example: Disjunctive cut

- \( \phi(r) = \|r\| \) is sublinear and \( C = \{ r : \phi(r) \leq 1 \} \)
- Intersection cut: \( 1 \leq \sum_j \phi(r^j) x_j \)
Example: Disjunctive cut

- $\phi(r) = \|r\|$ is sublinear and $C = \{r : \phi(r) \leq 1\}$
- Intersection cut: $1 \leq \sum_j \phi(r^j)x_j$

- Discussion: coefficient of $x_2$ doesn’t make much sense
An $S$-free set $C$ is **maximal** if there is no other $S$-free set that strictly contains $C$.
Example: Disjunctive cut

- Recall $\sum_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} x_j \in S := \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \}$

- $\phi(r) = \max\{ r_1, r_2 \}$ is sublinear and $C = \{ r : \phi(r) \leq 1 \}$ is maximal $S$-free
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- Intersection cut: $1 \leq \sum_j \phi \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} x_j \iff \sum_j \max\{\alpha_j, \beta_j\} x_j \geq 1$
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Discussion: coefficients make sense
Summary: Intersection cuts

- $S$ closed, $0 \notin S$ and
  \[ \sum r^j x_j \in S \]
  \[ x_j \geq 0 \]

- $C$ convex, $S$-free, $0 \in \text{int } C$, and $C = \{ x : \phi(x) \leq 1 \}$ with $\phi$ sublinear

- Intersection cut:
  \[ 1 \leq \sum_j \phi(r^j)x_j \]

- How can we strengthen the cut?
Modifying the rays

• From

\[ \sum r^j x_j \in S \]
\[ x_j \geq 0 \]

to

\[ \sum (r^j + m^j) x_j \in S + \sum m^i x_j \subseteq S + \sum m^i \text{dom}(x_j) \]
\[ x_j \geq 0 \]

• If \( C \) is \( (S + \sum m^i \text{dom}(x_j)) \)-free, we can build a possibly stronger cut:

\[ \sum \phi(r^i + m^i) x_j \geq 1 \]
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- The technique is called "monoidal strengthening" because \( \sum m^i \text{dom}(x_j) \) is a **monoid**: a set closed under addition, with an identity element.
• If $M$ is a monoid such that $x_j M \subseteq M$ for all $x_j$ in its domain, then we can obtain the relation

$$x_j \geq 0 \text{ and } \sum_j (r^j + m^j)x_j \in S + M.$$
• If $M$ is a monoid such that $x_j M \subseteq M$ for all $x_j$ in its domain, then we can obtain the relation

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• If, in addition, $C$ is also $(S + M)$-free, then we can get the cut

$$\sum_j \phi(r^j + m^j)x_j \geq 1$$
• If $M$ is a monoid such that $x_j M \subseteq M$ for all $x_j$ in its domain, then we can obtain the relation

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• If, in addition, $C$ is also $(S + M)$-free, then we can get the cut

$$\sum_j \phi(r_j^i + m_j^i) x_j \geq 1$$

• The best cut we can obtain with this procedure is

$$\sum_j \left( \inf_{m \in M} \phi(r_j^i + m) \right) x_j \geq 1$$
Observations

Proposition
If $M$ is a monoid, $C$ is $(S + M)$-free, and $\text{dom}(x_j)M \subseteq M$, then

$$\sum_j \left( \inf_{m \in M} \phi(r^j + m) \right) x_j \geq 1,$$

is valid

- The objective is to find a monoid such that
  - $\text{dom}(x_j)M \subseteq M$ (because then we can modify the rays and still be in $S + M$)
  - $C$ is $(S + M)$-free (because then we can still use $\phi$ to build the cut)

- $C$ is $(S + M)$-free $\iff C - M$ is $S$-free

- To build $M$ we ask ourselves, how can we move $C$ keeping it $S$-free?
Continuous variables \( \text{dom}(x_j) = \mathbb{R}_+ \)

- \( x_j \geq 0 \) and \( \left( \begin{array}{c} 1 \\ -3 \end{array} \right) x_1 + \left( \begin{array}{c} -2 \\ -\frac{1}{2} \end{array} \right) x_2 + \left( \begin{array}{c} 1 \\ 2 \end{array} \right) x_3 \in S \)

- \( x_j \geq 0 \) and \( \left( \begin{array}{c} 1 \\ -3 \end{array} \right) x_1 + \left( \left( \begin{array}{c} -2 \\ -\frac{1}{2} \end{array} \right) + \left( \begin{array}{c} 2 \\ \frac{1}{2} \end{array} \right) \right) x_2 + \left( \begin{array}{c} 1 \\ 2 \end{array} \right) x_3 \in S + \left( \begin{array}{c} 2 \\ \frac{1}{2} \end{array} \right) \mathbb{R}_+ \)

- Is \( C, S + \left( \begin{array}{c} 2 \\ \frac{1}{2} \end{array} \right) \mathbb{R}_+ \) -free?

\[ \begin{align*}
\phi(r^1) &= \sqrt{10} \\
\phi(r^2) &= \frac{\sqrt{17}}{2} \\
\phi(r^3) &= \sqrt{5}
\end{align*} \]
Continuous variables $(\text{dom}(x_j) = \mathbb{R}_+)$

- $x_j \geq 0$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix} x_1 + \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_3 \in S$

- $x_j \geq 0$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix} x_1 + \left(\begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}\right) x_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_3 \in S + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$

- Is $C$, $S + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$-free? Is $C - \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$, $S$-free?
Continuous variables \((\text{dom}(x_j) = \mathbb{R}_+)\)

- Applying monoidal strengthening to a continuous variable is only possible when \(C\) is not maximal.
- From now on, we only consider \(\text{dom}(x_j) = \mathbb{Z}_+\).
- Then, we can drop the requirement \(\text{dom}(x_j)M \subseteq M\), since \(\mathbb{Z}_+M = M\).

**Proposition (Balas, Jeroslow)**

If \(M\) is a monoid, \(C\) is \((S + M)\)-free, and \(x_j \in \mathbb{Z}_+\), then

\[
\sum_j \left( \inf_{m \in M} \phi(r^j + m) \right) x_j \geq 1, \text{ is valid}
\]
Integer variables. Example: One row relaxation

- Consider

\[ \frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in S := \mathbb{Z} - \frac{1}{3} \]

\[ x_1, x_2, x_3 \in \mathbb{Z} \]

\[ x_j \geq 0 \]

- Maximal $S$-free set $C = [-\frac{1}{3}, \frac{2}{3}]$

- How can we move $C$ and keep it $S$-free?
Integer variables. Example: One row relaxation

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\frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in S := \mathbb{Z} - \frac{1}{3}
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\[x_1, x_2 \in \mathbb{Z}\]

\[x_j \geq 0\]

• Maximal S-free set \( C = [-\frac{1}{3}, \frac{2}{3}] \)

• How can we move \( C \) and keep it S-free?

• Move it by \( \mathbb{Z} \). So the monoid is \(-\mathbb{Z} = \mathbb{Z}\)
Example (cont.): $\frac{5}{2} x_1 - \frac{1}{2} x_2 - x_3 \in \mathbb{Z} - \frac{1}{3}$

- Cut before monoidal strengthening: $\frac{15}{4} x_1 + \frac{3}{2} x_2 + 3 x_3 \geq 1$
- Cut after monoidal strengthening: $\frac{3}{4} x_1 + \frac{3}{4} x_2 + 3 x_3 \geq 1$
Example (cont.): $\frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in \mathbb{Z} - \frac{1}{3}$

- Cut before monoidal strengthening: $\frac{15}{4}x_1 + \frac{3}{2}x_2 + 3x_3 \geq 1$
- Cut after monoidal strengthening: $\frac{3}{4}x_1 + \frac{3}{4}x_2 + 3x_3 \geq 1$
- Why did the cut get better?
- MS modifies $r^1$ from $\frac{5}{2}$ to $\frac{1}{2}$ $\implies$ harder to leave $C$
Example (cont.): $\frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in \mathbb{Z} - \frac{1}{3}$

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MS modifies $r^2$ from $-\frac{1}{2}$ to $\frac{1}{2}$ $\implies$ harder to leave $C$
Consider again $S = \{y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1\}$ and $C = \{y : y_1 \leq 1, y_2 \leq 1\}$.

How can we move $C$ and keep it $S$-free?

Proposition

If $M \subseteq \neg \text{rec}(C)$, then $\inf_{m \in M} \phi(r + m) = \phi(r)$. Hence, monoidal strengthening does not strengthen.
Consider again $S = \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1\}$ and $C = \{ y : y_1 \leq 1, y_2 \leq 1\}$.

How can we move $C$ and keep it $S$-free?

The only way is to move it towards $\text{rec } C$, so the monoid would be $- \text{rec } C$.

This is exactly what we do not want.
Back to disjunctive cut

- Consider again $S = \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \}$ and $C = \{ y : y_1 \leq 1, y_2 \leq 1 \}$
- How can we move $C$ and keep it $S$-free?
- The only way is to move it towards $\text{rec } C$, so the monoid would be $\text{rec } C$
- This is exactly what we do not want

**Proposition**

If $M \subseteq \text{rec}(C)$, then $\inf_{m \in M} \phi(r + m) = \phi(r)$. Hence, monoidal strengthening does not strengthen.
Other reasons for why MS cannot work

\[ S = \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \} \]
\[ C = \{ y : y_1 \leq 1, y_2 \leq 1 \} \]
\[ M = - \text{rec}(C) \]

- The monoid is “continuous” (is a convex cone)
- \( S \) is a (translated) cone
Making monoidal strengthening work for disjunctive cut

- $S = \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1 \}$
- To make some room for $C$ to move we need to truncate $S$

$$S := \{ y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1, y_i \geq b_i \}$$
Making monoidal strengthening work for disjunctive cut

- $S = \{y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1\}$
- To make some room for $C$ to move we need to truncate $S$

$$S := \{y \in \mathbb{R}^2 : y_1 \geq 1 \lor y_2 \geq 1, y_i \geq b_i\}$$
• The conditions $y_i \geq b_i$ mean that our disjunctive terms $\sum_j \alpha_j x_j \geq 1$ and $\sum_j \beta_j x_j \geq 1$ become valid inequalities when relaxing the right-hand side to $b_1$ and $b_2$, resp.

• This is a non-trivial condition as it cannot always be achieved!
Consider now \( S = \{ (x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq -1 \} \)
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Monoidal strengthening for quadratic programming

More general:

- $S = \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|, a^T x + d^T y = -1\}$
- $C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} : \|y\| \leq \lambda^T x, a^T x + d^T y = -1\}$ with $\|\lambda\| = 1$.

**Theorem**

Let $(x_0, 0)$ be the “apex” of $C_\lambda$. If $\|d\| < -a^T \lambda$, then $M$ defined by

$$
\{0, 0\} \cup \{(x, y) \in \langle \{\lambda, a\} \rangle \times \mathbb{R}^m : a^T x + d^T y = 0, \|x - x_0\| \geq \|y\|, (a - \lambda^T a\lambda)^T x \leq -1\}
$$

is a monoid and $C - M$ is $S$-free.
Conclusions

- Monoidal strengthening is a technique for strengthening intersection cuts
- Continuous monoidal strengthening is related to maximal $S$-free sets
- When looking for a monoid we need to ask how to move $C$ while keeping it $S$-free and avoiding $-\text{rec}(C)$

Current and future work

- Implementation of monoidal strengthening for QP in SCIP (and COPT)
- Investigate monoidal strengthening for semi-continuous variables