Monoidal Strengthening

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joint work with Sven Wiese, Gonzalo Muñoz, Antonia Chmiela

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What is Monoidal Strengthening?

- Introduced by Balas and Jeroslow in 1980
- It is a technique for strengthening intersection cuts by exploiting the integrality of non-basic integer variables

Intersection cuts

- We can deduce conditions over the feasible set of a MIP of the form

$$\begin{aligned} \sum r^j x_j &\in S \\ x_j &\geq 0 \end{aligned}$$

  where S is closed and $0 \notin S$

Example: Disjunctive cut

- Consider the constraints $x_j \geq 0$ and the disjunction

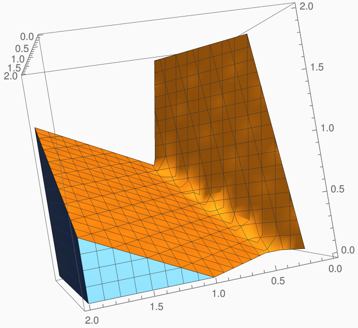
$$\sum_j \alpha_j x_j \geq 1$$
$$\vee \sum_j \beta_j x_j \geq 1$$

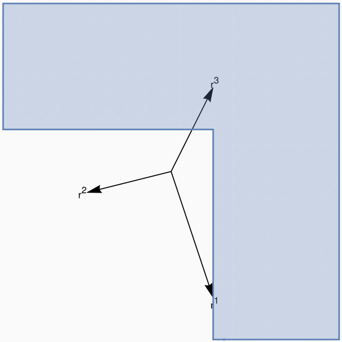
- This is equivalent to $x_j \geq 0$ and

$$\sum_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} x_j \in S := \{y \in \mathbb{R}^2 \,:\, y_1 \geq 1 \vee y_2 \geq 1\}$$

Example: Disjunctive cut

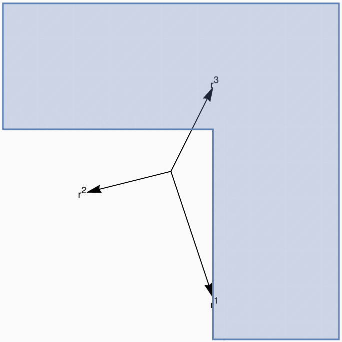
- Let $\alpha = (1, -2, 1)$ and $\beta = (-3, -\frac{1}{2}, 2)$
- Disjunction $\sum_j \alpha_j x_j \geq 1 \vee \sum_j \beta_j x_j \geq 1$
- Equivalently $\sum_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} x_j \in S := \{y \in \mathbb{R}^2 \ : \ y_1 \geq 1 \vee y_2 \geq 1\}$





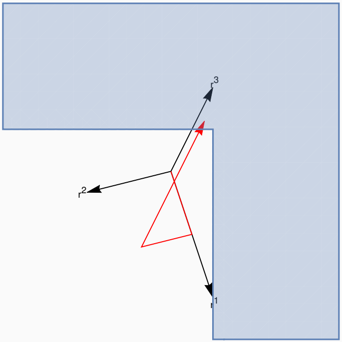
Example: Disjunctive cut

- A feasible solution must describe a path into S



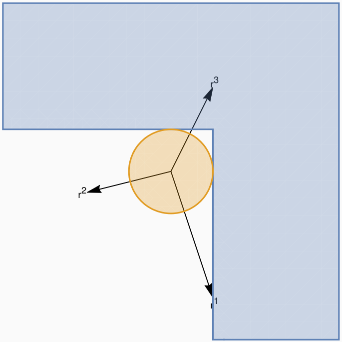
Example: Disjunctive cut

- A feasible solution must describe a path into S
- For example, $x = (\frac{1}{2}, \frac{3}{5}, \frac{3}{2})$



S-free set

- A convex set C is an S-free set if it does not contain any point of S in its interior
- Thus, if x is going to be feasible for $\begin{pmatrix} \sum r^j x_j \in S \\ x_j \geq 0 \end{pmatrix}$, it better be that $x_j \geq 0$ and $\sum r^j x_j \notin \text{int}(C)$



Representations of S-free set and intersection cuts

- ϕ is a sublinear if it is
  - positively homogeneous: $\phi(rx) = \phi(r)x$ for all $x \geq 0$
  - subadditive: $\phi(\sum_j r^j) \leq \sum_j \phi(r^j)$
- As C is convex with 0 in its interior, there is ϕ sublinear such that

$$C = \{x : \phi(x) \leq 1\}$$
$$(\text{int } C)^c = \{x : \phi(x) \geq 1\}$$

- If $x_j \geq 0$ and $\sum_j r^j x_j \notin \text{int}(C)$, then

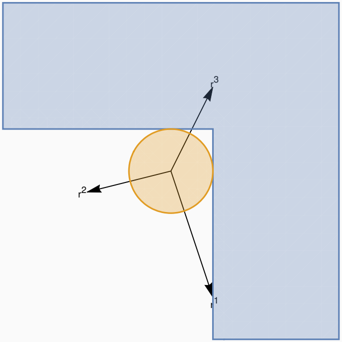
$$1 \leq \phi(\sum_j r^j x_j) \leq \sum_j \phi(r^j x_j) = \sum_j \phi(r^j) x_j$$

- Intersection cut:

$$1 \leq \sum_j \phi(r^j) x_j$$

Example: Disjunctive cut

- $\phi(r) = \|r\|$ is sublinear and $C = \{r : \phi(r) \leq 1\}$



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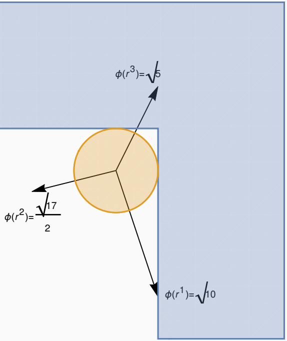

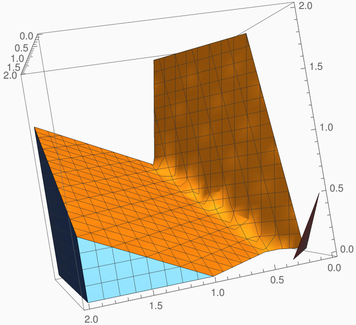

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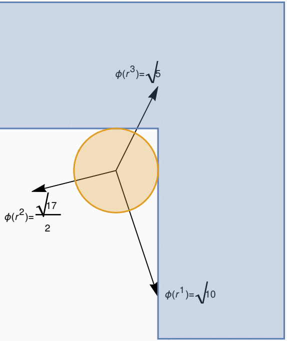
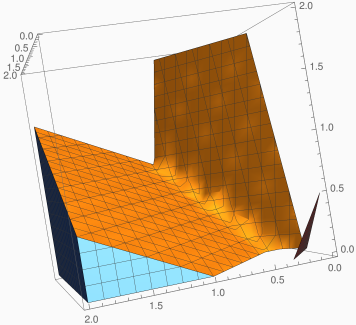
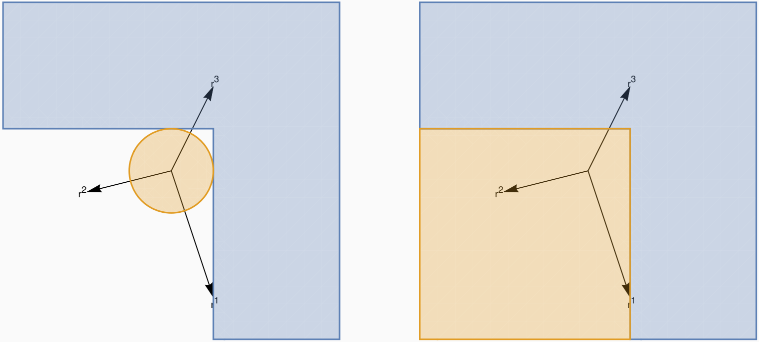




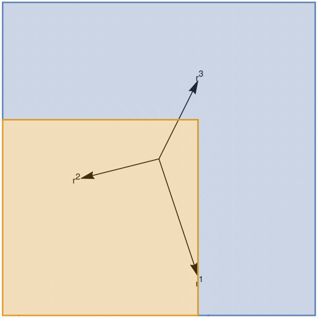
- Discussion: coefficient of x_2 doesn't make much sense

- An S-free set C is maximal if there is no other S-free set that strictly contains C



Example: Disjunctive cut

- Recall $\sum_j \begin{pmatrix}\alpha_j \\ \beta_j\end{pmatrix} x_j \in S := \{y \in \mathbb{R}^2 : y_1 \geq 1 \vee y_2 \geq 1\}$
- $\phi(r) = \max\{r_1, r_2\}$ is sublinear and $C = \{r : \phi(r) \leq 1\}$ is maximal S-free



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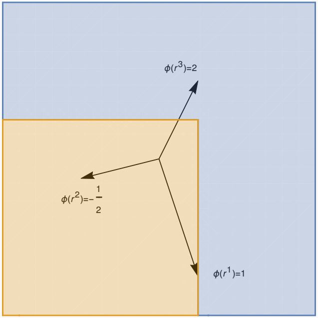
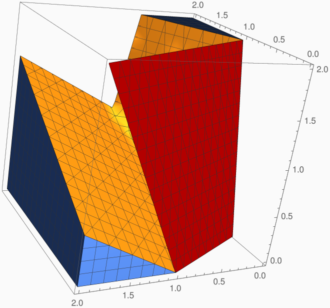
- Intersection cut: $1 \leq \sum_j \phi \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} x_j \iff \sum_j \max\{\alpha_j, \beta_j\} x_j \geq 1$





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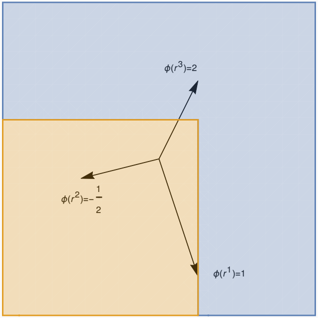

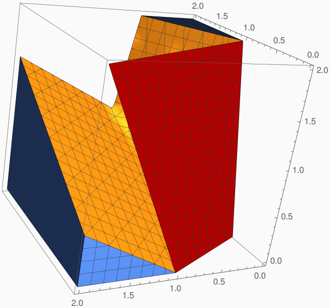

- Discussion: coefficients make sense

Summary: Intersection cuts

- S closed, $0 \notin S$ and

$$\sum r^j x_j \in S$$
$$x_j \geq 0$$

- C convex, S-free, $0 \in \operatorname{int} C$, and $C = \{x : \phi(x) \leq 1\}$ with ϕ sublinear
- Intersection cut:

$$1 \leq \sum_j \phi(r^j) x_j$$

- How can we strengthen the cut?

Modifying the rays

- From

$$\sum r^j x_j \in S$$
$$x_j \geq 0$$

to

$$\sum (r^j + m^j) x_j \in S + \sum m^j x_j \subseteq S + \sum m^j \operatorname{dom}(x_j)$$
$$x_j \geq 0$$

- If C is $(S + \sum m^j \operatorname{dom}(x_j))$-free, we can build a possibly stronger cut:

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- The technique is called "monoidal strengthening" because $\sum m^j \operatorname{dom}(x_j)$ is a monoid: a set closed under addition, with an identity element.

Monoidal Strengthening

- If M is a monoid such that $x_j M \subseteq M$ for all x_j in its domain, then we can obtain the relation

$$x_j \geq 0 \text{ and } \sum_j (r^j + m^j) x_j \in S + M.$$

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- The best cut we can obtain with this procedure is

$$\sum_j \left(\inf_{m \in M} \phi(r^j + m) \right) x_j \geq 1$$

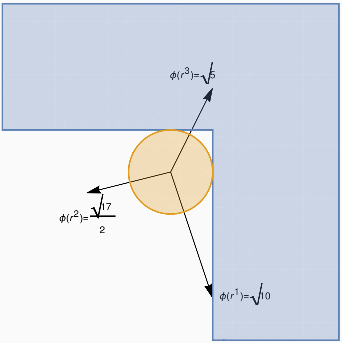
Proposition
If M is a monoid, C is $(S+M)$-free, and $\mathrm{dom}(x_j)M \subseteq M$*, then*

$$\sum_j \left(\inf_{m \in M} \phi(r^j + m) \right) x_j \geq 1, \text{ is valid}$$

- The objective is to find a monoid such that
  - $\mathrm{dom}(x_j)M \subseteq M$ (because then we can modify the rays and still be in $S+M$)
  - C is $(S+M)$-free (because then we can still use ϕ to build the cut)
- C is $(S+M)$-free $\iff$ $C-M$ is S-free
- To build M we ask ourselves, how can we move C keeping it S-free?

Continuous variables ($\mathrm{dom}(x_j) = \mathbb{R}_+$)

- $x_j \geq 0$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix} x_1 + \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_3 \in S$
- $x_j \geq 0$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix} x_1 + \left(\begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \right) x_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_3 \in S + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$
- Is C, $S + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$ -free?



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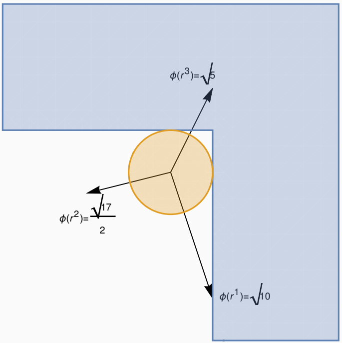
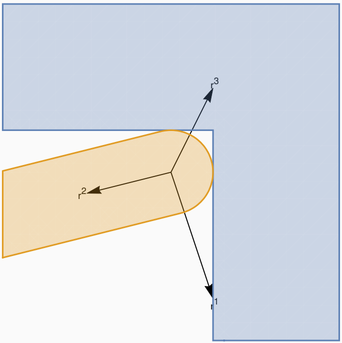
- Is C, $S + \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$ -free? Is $C - \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \mathbb{R}_+$, S-free?





Continuous variables ($\mathrm{dom}(x_j) = \mathbb{R}_+$)

- Applying monoidal strengthening to a continuous variable is only possible when C is not maximal
- From now on, we only consider $\mathrm{dom}(x_j) = \mathbb{Z}_+$
- Then, we can drop the requirement $\mathrm{dom}(x_j)M \subseteq M$, since $\mathbb{Z}_+ M = M$

Proposition (Balas, Jeroslow)
If M is a monoid, C is $(S + M)$-free, and $x_j \in \mathbb{Z}_+$, then

$$\sum_j \left(\inf_{m \in M} \phi(r^j + m) \right) x_j \geq 1, \text{ is valid}$$

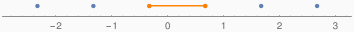
Integer variables. Example: One row relaxation

- Consider

$$\frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in S := \mathbb{Z} - \frac{1}{3}$$
$$x_1, x_2 \in \mathbb{Z}$$
$$x_j \geq 0$$

- Maximal S-free set $C = [-\frac{1}{3}, \frac{2}{3}]$

S



C

- How can we move C and keep it S-free?

- Consider

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$$x_1, x_2 \in \mathbb{Z}$$
$$x_j \geq 0$$

- Maximal S-free set $C = [-\frac{1}{3}, \frac{2}{3}]$

S

C

- How can we move C and keep it S-free?
- Move it by $\mathbb{Z}$. So the monoid is $-\mathbb{Z} = \mathbb{Z}$

Example (cont.): $\frac{5}{2}x_1 - \frac{1}{2}x_2 - x_3 \in \mathbb{Z} - \frac{1}{3}$

- Cut before monoidal strengthening: $\frac{15}{4}x_1 + \frac{3}{2}x_2 + 3x_3 \geq 1$
- Cut after monoidal strengthening: $\frac{3}{4}x_1 + \frac{3}{4}x_2 + 3x_3 \geq 1$

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- Why did the cut get better?
- MS modifies r^1 from $\frac{5}{2}$ to $\frac{1}{2}$ $\implies$ harder to leave C

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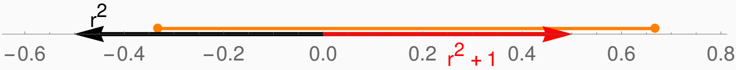
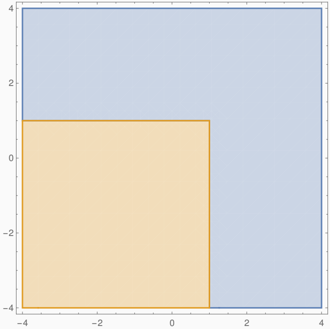
- Why did the cut get better?
- MS modifies r^1 from $\frac{5}{2}$ to $\frac{1}{2}$ $\Longrightarrow$ harder to leave C

- MS modifies r^2 from $-\frac{1}{2}$ to $\frac{1}{2}$ $\Longrightarrow$ harder to leave C



Back to disjunctive cut

- Consider again $S = \{y \in \mathbb{R}^2 \,:\, y_1 \geq 1 \vee y_2 \geq 1\}$ and $C = \{y \,:\, y_1 \leq 1, y_2 \leq 1\}$
- How can we move C and keep it S-free?



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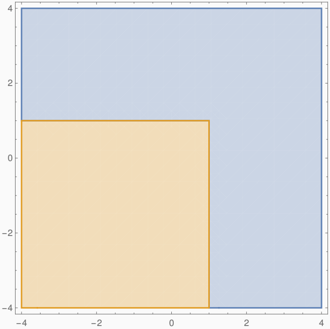
- Consider again $S = \{y \in \mathbb{R}^2 : y_1 \geq 1 \vee y_2 \geq 1\}$ and $C = \{y : y_1 \leq 1, y_2 \leq 1\}$
- How can we move C and keep it S-free?
- The only way is to move it towards $\operatorname{rec} C$, so the monoid would be $-\operatorname{rec} C$
- This is exactly what we do not want



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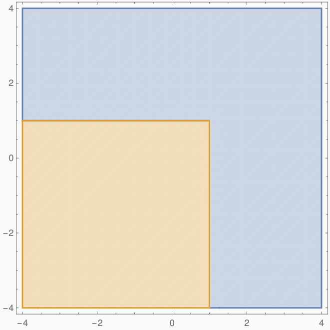
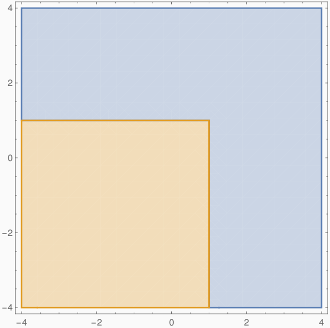


Proposition
If $M \subseteq -\operatorname{rec}(C)$, then $\inf_{m \in M} \phi(r+m) = \phi(r)$. Hence, monoidal strengthening does not strengthen.

Other reasons for why MS cannot work

$$S = \{y \in \mathbb{R}^2 : y_1 \geq 1 \vee y_2 \geq 1\}$$
$$C = \{y : y_1 \leq 1, y_2 \leq 1\}$$
$$M = -\operatorname{rec}(C)$$

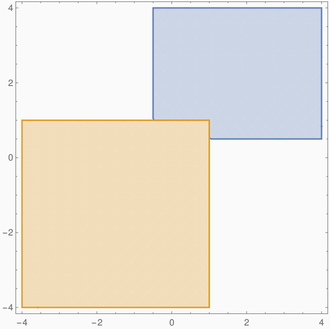
- The monoid is "continuous" (is a convex cone)
- S is a (translated) cone



Making monoidal strengthening work for disjunctive cut

- $S = \{y \in \mathbb{R}^2 : y_1 \geq 1 \vee y_2 \geq 1\}$
- To make some room for C to move we need to truncate S

$$S := \{y \in \mathbb{R}^2 : y_1 \geq 1 \vee y_2 \geq 1, y_i \geq b_i\}$$



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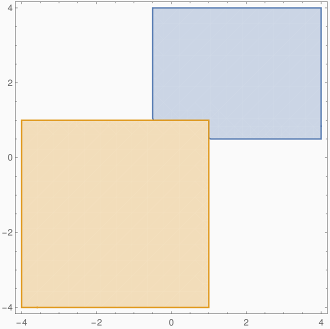
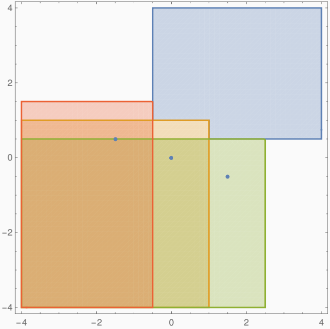
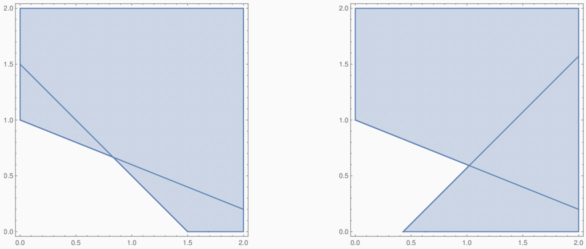




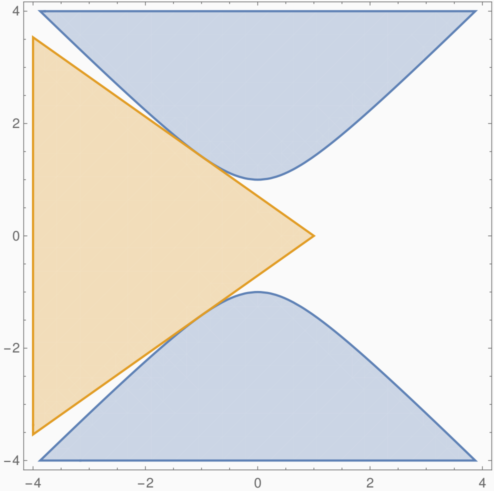
Making monoidal strengthening work for disjunctive cut

- The conditions $y_i \geq b_i$ mean that our disjunctive terms $\sum_j \alpha_j x_j \geq 1$ and $\sum_j \beta_j x_j \geq 1$ become valid inequalities when relaxing the right-hand side to b_1 and b_2, resp.
- This is a non-trivial condition as it cannot always be achieved!



Monoidal strengthening for quadratic programming

- Consider now $S = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq -1\}$



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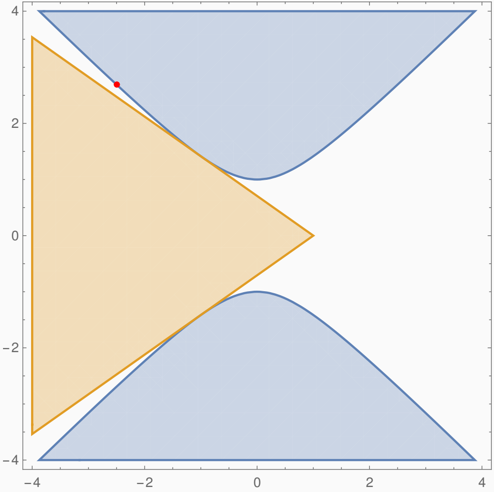

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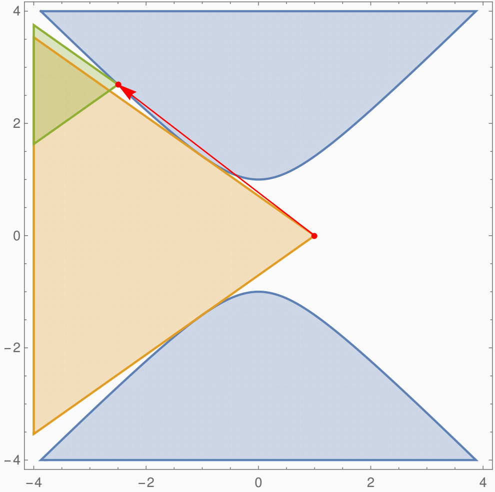

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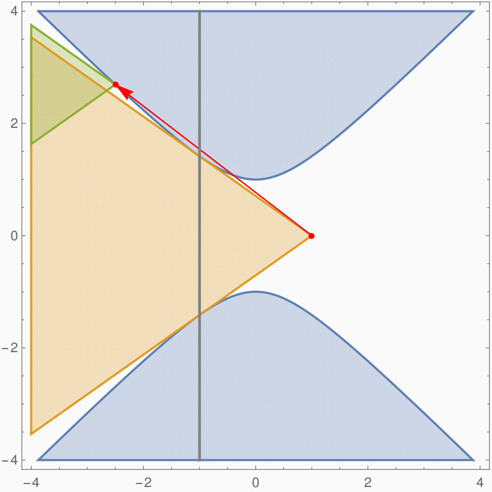


Monoidal strengthening for quadratic programming

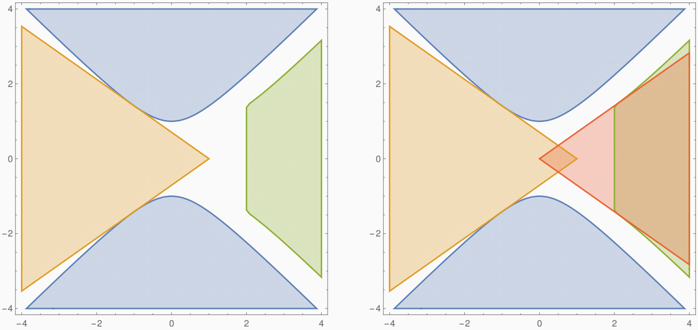
More general:

- $S = \{(x, y) \in \mathbb{R}^{n+m} \; : \; \|x\| \le \|y\|, \; a^\mathsf{T} x + d^\mathsf{T} y = -1\}$
- $C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} \; : \; \|y\| \le \lambda^\mathsf{T} x, \; a^\mathsf{T} x + d^\mathsf{T} y = -1\}$ with $\|\lambda\| = 1$.

Theorem
Let $(x_0, 0)$ be the "apex" of C_λ. If $\|d\| < -a^\mathsf{T}\lambda$, then M defined by

$$\{0,0\} \cup \{(x, y) \in \langle \{\lambda, a\} \rangle \times \mathbb{R}^m \; : \; a^\mathsf{T} x + d^\mathsf{T} y = 0, \|x - x_0\| \ge \|y\|, (a - \lambda^\mathsf{T} a \lambda)^\mathsf{T} x \le -1\}$$

is a monoid and $C - M$ is S-free.



Conclusions

- Monoidal strengthening is a technique for strengthening intersection cuts
- Continuous monoidal strengthening is related to maximal S-free sets
- When looking for a monoid we need to ask how to move C while keeping it S-free and avoiding $-\operatorname{rec}(C)$

Current and future work

- Implementation of monoidal strengthening for QP in SCIP (and COPT)
- Investigate monoidal strengthening for semi-continuous variables