On obtaining the convex hull of quadratic inequalities via aggregations

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Context
Quadratically Constrained Quadratic Program

**QCQP**
Quadratic objective, quadratic constraints:

\[
\begin{align*}
\text{min} & \quad x^T Q_0 x + b_0^T x \\
\text{s.t.} & \quad x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [m]
\end{align*}
\]
Quadratically Constrained Quadratic Program

**QCQP**
May be equivalently written as:

$$\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [m]
\end{align*}$$
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\end{align*}
\]

- Thus, we care about

\[
\text{conv} \left\{ x \mid x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [m] \right\}
\]
**QCQP**
May be equivalently written as:

$$\begin{align*}
\text{min} & \quad c^\top x \\
\text{s.t.} & \quad x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [m]
\end{align*}$$

- Thus, we care about

$$\text{conv} \{ x \mid x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [m] \}$$

- Challenging to compute! So we can consider “partial” convexifications
Two-row relaxations

- Single rows are not really useful to convexify.
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- We can select two rows and try to find the convex hull of their intersection:

\[ C_2 = \left\{ x \in \mathbb{R}^n \mid x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [2] \right\} \]
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  \[ C_2 = \{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [2] \} \]
- For some technical reasons, we consider the “open version” of the above set:
  \[ O_2 = \{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x < d_i \quad \forall i \in [2] \} \]
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- For some technical reasons, we consider the “open version” of the above set:

\[ \mathcal{O}_2 = \left\{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x < d_i \ \forall i \in [2] \right\} \]

- It turns out the convex hull of \( \mathcal{O}_2 \) is well understood!
Let’s first talk about aggregations

Given $\lambda \in \mathbb{R}^m$ and

$$S := \{ x \mid x^T Q_i x + b_i^T x < d_i \ \forall i \in [m] \} ,$$
Let’s first talk about aggregations

Given \( \lambda \in \mathbb{R}^m_+ \) and

\[
S := \{ x \mid x^T Q_i x + b_i^T x < d_i \quad \forall i \in [m] \},
\]

\[
S^\lambda := \left\{ x \mid x^T \left( \sum_{i=1}^m \lambda_i Q_i \right) x + \left( \sum_{i=1}^m \lambda_i b_i \right)^T x < \left( \sum_{i=1}^m \lambda_i d_i \right) \quad \forall i \in [m] \right\}
\]

is a relaxation of \( S \).

We are multiplying \( i^{th} \) constraint by \( \lambda_i \) and then adding them together.
Convex hull of $\mathcal{O}_2$

$$\mathcal{O}_2 = \left\{ x \in \mathbb{R}^n \mid x^T Q_i x + b_i^T x < d_i \; \forall i \in [2] \right\}$$
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**Theorem (Yildiran (2009))**

Given a set $\mathcal{O}_2$, such that $\text{conv}(\mathcal{O}_2) \neq \mathbb{R}^n$, there exists $\lambda^1, \lambda^2 \in \mathbb{R}_+^2$ such that:

$$\text{conv}(\mathcal{O}_2) = (\mathcal{O}_2)^{\lambda^1} \cap (\mathcal{O}_2)^{\lambda^2}.$$
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- Yildiran (2009) also gives an algorithm to compute $\lambda_1$ and $\lambda_2$.
- The quadratic constraints in $(\mathcal{O}_2)^{\lambda^i}$ $i \in \{1, 2\}$ have very nice properties:
  - $\sum_{j=1}^2 \lambda_j^i Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$
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  - $\sum_{j=1}^{2} \lambda^i_j Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$
  - Basically, the sets $(\mathcal{O}_2)^{\lambda^i}$ $i \in \{1, 2\}$ are either **ellipsoids** or **hyperboloids** (union of two convex sets).
  - Henceforth, we call a quadratic constraint with the “quadratic part” having at most one negative eigenvalue a **good constraint**.
Example

\[ S := \left\{ x, y \mid -xy < -1, \quad x^2 + y^2 < 9 \right\} \]
Example

\[ \text{conv}(S) := \left\{ x, y \mid \frac{(x - y)^2}{7} < 7, \frac{x^2 + y^2}{9} < 9 \right\} \]
Example

\[ S := \left\{ x, y \mid -xy < -1 \quad x^2 + y^2 < 9 \right\} \]

\[ \text{conv}(S) := \left\{ x, y \mid (x - y)^2 < 7 \quad x^2 + y^2 < 9 \right\} \]

With the blue quadratic coming from \( \lambda^1 = (2, 1) \)

\[
\begin{align*}
-xy &< -1 \cdot 2 \\
+ x^2 + y^2 &< 9 \cdot 1
\end{align*}
\]
Example

\[ S := \left\{ x, y \mid \begin{array}{c} -xy < -1 \\ x^2 + y^2 < 9 \end{array} \right\} \]

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With the blue quadratic coming from \( \lambda^1 = (2, 1) \)

\[ -xy < -1 \cdot 2 \\
+ \quad x^2 + y^2 < 9 \cdot 1 \]

\[ x^2 - 2xy + y^2 < 7 \quad \equiv (x - y)^2 < 7 \]
Literature survey

Related results:

- [Yildiran (2009)]
- [Burer and Kılınç-Karzan (2017)] (second order cone intersected with a nonconvex quadratic)
- [Modaresi and Vielma (2017)] (closed version of results)
Literature survey

Related results:

- [Yildiran (2009)]
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Other related papers:

- [Tawarmalani, Richard, Chung (2010)] (covering bilinear knapsack)
- [Santana and Dey (2020)] (polytope and one quadratic constraint)
- [Ye and Zhang (2003)], [Burer and Anstreicher (2013)], [Bienstock (2014)]
- [Bienstock, Chen, and Muñoz (2020)], [Muñoz and Serrano (2020)] (cuts for QCQP using intersection cuts approach)
- ...
The question we consider...

We want to understand the power of aggregations for $m \geq 3$
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**Main contribution**
Under some technical conditions, intersection of aggregations yield the convex hull for three quadratic constraints.
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**Additional contribution**
The above result represents the limit of aggregations.
The question we consider...

We want to understand the power of aggregations for \( m \geq 3 \)

**Main contribution**
Under some technical conditions, intersection of aggregations yield the convex hull for three quadratic constraints.

**Additional contribution**
The above result represents the limit of aggregations. Basically, aggregations \( \not\rightarrow \) convex hull if the technical sufficient condition does not hold for \( m = 3 \) or when \( m \geq 4 \).
Main results
Theorem

Let $n \geq 3$ and

$$O_3 = \left\{ x \in \mathbb{R}^n \mid \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, \ i \in [3] \right\}.$$
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$$

Assume

- (PDLC) There exists $\theta \in \mathbb{R}^3$ such that $\sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \succ 0$.
- (Non-trivial convex hull) $\text{conv}(O_3) \neq \mathbb{R}^n$. 

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- **(Non-trivial convex hull)** $\text{conv}(O_3) \neq \mathbb{R}^n.$

Let $\Omega := \left\{ \lambda \in \mathbb{R}_+^3 \mid (O_3)^\lambda \supseteq \text{conv}(O_3) \text{ and } (O_3)^\lambda \text{ is good} \right\},$
Theorem

Let $n \geq 3$ and

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$$\text{conv}(O_3) = \bigcap_{\lambda \in \Omega} (O_3)^\lambda.$$
Example

\[ S := \left\{ (x, y, z) \mid \begin{array}{c}
x^2 + y^2 < 2 \\
-x^2 - y^2 < -1 \\
-x^2 + y^2 + z^2 + 6x < 0
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Example

\[ S := \left\{ (x, y, z) \middle| \begin{array}{l} x^2 + y^2 < 2 \\ -x^2 - y^2 < -1 \\ -x^2 + y^2 + z^2 + 6x < 0 \end{array} \right\} \]

\[ \text{conv}(S) := \left\{ (x, y, z) \middle| \begin{array}{l} x^2 + y^2 < 2 \\ -2x^2 + z^2 + 6x < -1 \\ -x^2 + y^2 + z^2 + 6x < 0 \end{array} \right\} \]
### Comparison of Results

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When does it hold? $\text{conv}(S) \neq \mathbb{R}^n$ PDLC condition, $\text{conv}(S) \neq \mathbb{R}^n$ How many aggregated inequalities needed? $2^\infty$ (Conjecture!) Structure of aggregated inequalities Polynomial-time algorithm exists to find them Even checking if $\lambda \in \Omega$ is not clear.
## Comparsion of results

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The closed case

**Theorem**

Let \( n \geq 3 \) and let

\[
C_3 = \left\{ x \in \mathbb{R}^n \mid \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} A_i & b_i \ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \ i \in [3] \right\}.
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- (Non-trivial convex hull) \( \text{conv}(C_3) \neq \mathbb{R}^n \).
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- (Non-trivial convex hull) $\text{conv}(C_3) \neq \mathbb{R}^n$.
- (No low-dimensional components) $C_3 \subseteq \text{int}(C_3)$.

Let $\Omega := \left\{ \lambda \in \mathbb{R}_+^3 \mid (C_3)^\lambda \supseteq \text{conv}(C_3) \right\}$,
The closed case

**Theorem**

Let $n \geq 3$ and let

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$$\overline{\text{conv}(C_3)} = \bigcap_{\lambda \in \Omega} (C_3)^\lambda.$$
Counterexamples
$m = 3$ but not satisfying PDLC condition

\[ S := \left\{ (x, y, z) \ \middle| \ \begin{array}{c} x^2 < 1 \\ y^2 < 1 \\ -xy + z^2 < 0 \end{array} \right\} \]

- PDLC condition does not hold, \( \text{conv}(S) \neq \mathbb{R}^3 \)
$m = 3$ but not satisfying PDLC condition

$S := \left\{ (x, y, z) \mid \begin{array}{l}
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\end{array} \right\}$

- PDLC condition does not hold, $\text{conv}(S) \neq \mathbb{R}^3$
$m = 4$ and satisfying PDLC

$$S := \left\{ (x, y, z) \mid \begin{align*}
x^2 + y^2 + z^2 + 2.2(xy + yz + xz) &< 1 \\
-2.1x^2 + y^2 + z^2 &< 0 \\
x^2 - 2.1y^2 + z^2 &< 0 \\
x^2 + y^2 - 2.1z^2 &< 0
\end{align*} \right\}$$

- PDLC condition holds, $\text{conv}(S) \neq \mathbb{R}^3$
m = 4 and satisfying PDLC

\[ S := \begin{cases} (x, y, z) \\ x^2 + y^2 + z^2 + 2.2(xy + yz + xz) < 1 \\ -2.1x^2 + y^2 + z^2 < 0 \\ x^2 - 2.1y^2 + z^2 < 0 \\ x^2 + y^2 - 2.1z^2 < 0 \end{cases} \]

- PDLC condition holds, \( \text{conv}(S) \neq \mathbb{R}^3 \)

\[ \text{conv}(S) \neq \bigcap_{\lambda \in \Omega} S^\lambda \]
Do we need a finite number of aggregations?

A non-counterexample:

\[ S := \{ x, y \mid x^2 \leq 1, \ y^2 \leq 1, \ (x - 1)^2 + (y - 1)^2 \geq 1 \} , \]

- Let \( \Omega^+ := \{ \lambda \in \mathbb{R}^3_+ \mid S^\lambda \supseteq \text{conv}(S) \} \)
Do we need a finite number of aggregations?

A non-counterexample:

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- Let \( \Omega^+ := \{ \lambda \in \mathbb{R}_+^3 \mid S^\lambda \supseteq \text{conv}(S) \} \)
- \( \text{conv}(S) = \bigcap_{\lambda \in \Omega^+} S^\lambda \).
- \( \text{conv}(S) \subsetneq \bigcap_{\lambda \in \tilde{\Omega}^+} S^\lambda \) for any \( \tilde{\Omega}^+ \subseteq \Omega^+ \) which is finite.
Do we need a finite number of aggregations?

A non-counterexample:

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But PDLC does not hold!
Main proof outline
Lemma

Let $n \geq 3$ and let $g_1, g_2, g_3 : \mathbb{R}^n \to \mathbb{R}$ be homogeneous quadratic functions:

$$g_i(x) = x^\top Q_i x.$$

Assuming there is a linear combination of $Q_1, Q_2, Q_3$ that is positive definite, the following equivalence holds

$$\{ x \in \mathbb{R}^n : g_i(x) < 0, \ i \in [3] \} = \emptyset \iff \exists \lambda \in \mathbb{R}^3_+ \setminus \{0\}, \sum_{i=1}^{3} \lambda_i Q_i \succeq 0.$$
\[ \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \] proof idea

\[ \text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \] is straight-forward
\[ \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \] 

**proof idea**

\[
\text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \text{ is straight-forward}
\]

\[
\text{conv}(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda:
\]

- Pick \( x^* \in \mathbb{R}^n \) such that \( x^* \not\in \text{conv}(S) \). We want to show that it lies outside some aggregation.
- Separation theorem \( \Rightarrow \) there exists \( \alpha^T x < \beta \) valid for \( \text{conv}(S) \) that separates \( x^* \).

\[ S^\alpha \cap \alpha^T x = \beta \]
$$\text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda$$  proof idea

$\text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda$ is straight-forward

$\text{conv}(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda$:

- Pick $x^* \in \mathbb{R}^n$ such that $x^* \not\in \text{conv}(S)$. We want to show that is lies outside some aggregation
\[ \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \]

**proof idea**

\( \text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \) is straightforward

\( \text{conv}(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda \):

- Pick \( x^* \in \mathbb{R}^n \) such that \( x^* \not\in \text{conv}(S) \). We want to show that it lies outside some aggregation
- **Separation theorem** \( \Rightarrow \) there exists \( \alpha^\top x < \beta \) valid for \( \text{conv}(S) \) that separates \( x^* \).
\[
\text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda
\]

**proof idea**

- **(Homogenization)** The above can be shown to imply: \(\{x | \alpha^T x = \beta x_{n+1}\}\) (call it \(H\)) does not intersect homogenization of \(S\):

  \[
  H \cap \left\{ (x, x_{n+1}) \mid \left[ \begin{array}{cc} x & x_{n+1} \end{array} \right] \left[ \begin{array}{cc} A_i & b_i \b_T \end{array} \right] \left[ \begin{array}{c} x \\ x_{n+1} \end{array} \right] < 0, \ i \in [3] \right\} = \emptyset.
  \]
\( \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \) proof idea

- Applying S-lemma we obtain \( \lambda \in \Omega \) such that

\[
H \cap \left\{ (x, x_{n+1}) \mid [x \quad x_{n+1}] \left( \sum_{i=1}^{3} \lambda_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \right) \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \right\} = \emptyset.
\]
\[ \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \] proof idea

- **Dehomogenizing**, we obtain \( S^\lambda \supseteq \text{conv}(S) \) that excludes \( x^* \)
Summary and open questions

- We have shown that, under technical assumptions, aggregations are enough to describe the convex hull of 3 quadratics.
- We have also shown that the result is not true if some conditions are relaxed.

We do not know if Ω can be refined to a finite set.
We do not completely understand the PDLC condition. What is its geometrical meaning? Can we replace it by another condition and obtain a similar result?
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Thank you!