On obtaining the convex hull of quadratic inequalities via aggregations

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Context

Quadratically Constrained Quadratic Program

QCQP

Quadratic objective, quadratic constraints:

$$\begin{aligned} \min \quad & x^\top Q_0 x + b_0^\top x \\ \text{s.t.} \quad & x^\top Q_i x + b_i^\top x \leq d_i \ \forall i \in [m] \end{aligned}$$

Quadratically Constrained Quadratic Program

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May be equivalently written as:

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- Challenging to compute! So we can consider "partial" convexifications

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- For some technical reasons, we consider the "open version" of the above set:

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- It turns out the convex hull of $\mathcal{O}_2$ is well understood!

Let's first talk about aggregations

Given $\lambda \in \mathbb{R}^m_+$ and

$$S := \left\{ x \;\middle|\; x^\top Q_i x + b_i^\top x < \;\; d_i \;\forall i \in [m] \right\},$$

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$$S := \left\{ x \;\middle|\; x^\top Q_i x + b_i^\top x < \; d_i \; \forall i \in [m] \right\},$$

$$S^\lambda := \left\{ x \;\middle|\; x^\top \left(\sum_{i=1}^m \lambda_i Q_i \right) x + \left(\sum_{i=1}^m \lambda_i b_i \right)^\top x \;\; < \;\; \left(\sum_{i=1}^m \lambda_i d_i \right) \; \forall i \in [m] \right\}$$

is a relaxation of S.

We are multiplying i^{th} constraint by λ_i and then adding them together.

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Theorem (Yildiran (2009))
Given a set $\mathcal{O}_2$, such that $\operatorname{conv}(\mathcal{O}_2) \neq \mathbb{R}^n$*, there exists $\lambda^1, \lambda^2 \in \mathbb{R}^2_+$ such that:*

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- The quadratic constraints in $(\mathcal{O}_2)^{\lambda^i}$ $i \in \{1, 2\}$ have very nice properties:
  - $\sum_{j=1}^2 \lambda_j^i Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$

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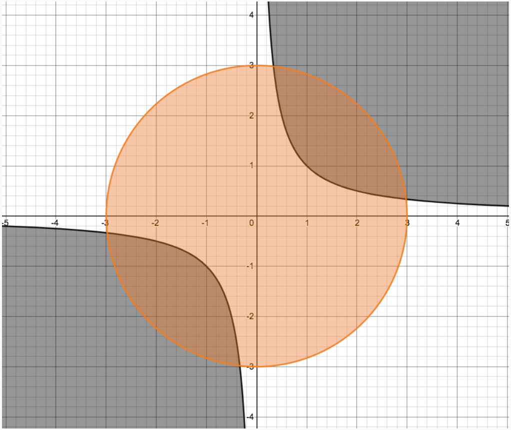
  - Basically, the sets $(\mathcal{O}2)^{\lambda_i}$ $i \in \{1, 2\}$ are either ellipsoids or hyperboloids (union of two convex sets).
  - Henceforth, we call a quadratic constraint with the "quadratic part" having at most one negative eigenvalue a good constraint.

Example

$$S := \left\{ x, y \;\middle|\; \begin{array}{rcl} -xy & < & -1 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



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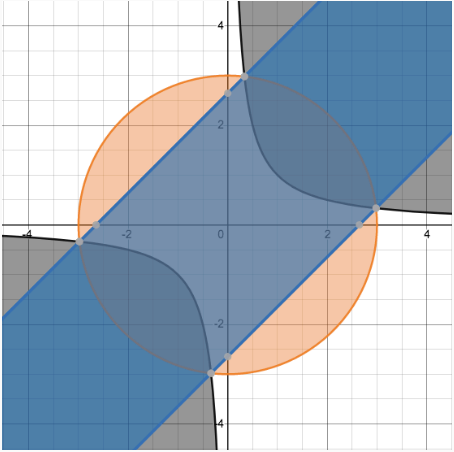
$$\text{conv}(S) := \left\{ x, y \;\middle|\; \begin{array}{rcl} (x-y)^2 & < & 7 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



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$$\begin{array}{lccrl} & -xy & < & -1 & \cdot 2 \\ + & x^2 + y^2 & < & 9 & \cdot 1 \\ \hline \end{array}$$

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$$\begin{array}{lrcrl} & -xy & < & -1 & \cdot 2 \\ + & x^2 + y^2 & < & 9 & \cdot 1 \\ \hline & x^2 - 2xy + y^2 & < & 7 & \equiv (x - y)^2 < 7 \end{array}$$

Literature survey

Related results:

- [Yildiran (2009)]
- [Burer and Kılınc-Karzan (2017)] (second order cone intersected with a nonconvex quadratic)
- [Modaresi and Vielma (2017)] (closed version of results)

Literature survey

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Other related papers:

- [Tawarmalani, Richard, Chung (2010)] (covering bilinear knapsack)
- [Santana and Dey (2020)] (polytope and one quadratic constraint)
- [Ye and Zhang (2003)], [Burer and Anstreicher (2013)], [Bienstock (2014)] [Burer (2015)], [Burer and Yang (2015)], [Anstreicher (2017)] (extended trust-region problem)
- [Burer and Ye (2019)], [Wang and Kılınc-Karzan (2020, 2021)], [Argue, Kılınc-Karzan, and Wang (2020)] (general conditions for the SDP relaxation being tight)
- [Bienstock, Chen, and Muñoz (2020)], [Muñoz and Serrano (2020)] (cuts for QCQP using intersection cuts approach)
- ...

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We want to understand the power of aggregations for $m \geq 3$

Main contribution
Under some technical conditions, intersection of aggregations yield the convex hull for <u>three</u> quadratic constraints.

Additional contribution
The above result represents the limit of aggregations. Basically, aggregations $\not\to$ convex hull if the technical suffcient condition does not hold for $m = 3$ or when $m \geq 4$.

Main results

Three rows: main result

Theorem

Let $n \geq 3$ *and*

$$\mathcal{O}_3 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0,\; i \in [3] \right\}.$$

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Assume

- *(PDLC) There exists $\theta \in \mathbb{R}^3$ such that $\sum_{i=1}^3 \theta_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \succ 0$.*
- *(Non-trivial convex hull) $\text{conv}(\mathcal{O}_3) \neq \mathbb{R}^n$.*

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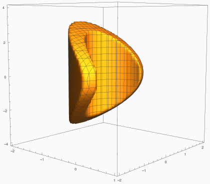
Example

$$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcl} x^2+y^2 & < & 2 \\ -x^2-y^2 & < & -1 \\ -x^2+y^2+z^2+6x & < & 0 \end{array} \right\}$$



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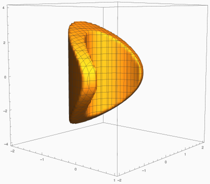
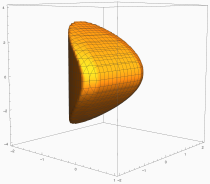


$$\mathrm{conv}(S) := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcl} x^2 + y^2 & < & 2 \\ -2x^2 + z^2 + 6x & < & -1 \\ -x^2 + y^2 + z^2 + 6x & < & 0 \end{array} \right\}$$



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How many aggregated inequalities needed?	2	∞ (Conjecture!)
Structure of aggregated inequalities	Polynomial-time algorithm exists to find them	Even checking if $\lambda \in \Omega$ is not clear.

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- *(Non-trivial convex hull)* $\mathrm{conv}(\mathcal{C}_3) \neq \mathbb{R}^n$.
- *(No low-dimensional components)* $\mathcal{C}_3 \subseteq \overline{\mathrm{int}(\mathcal{C}_3)}$.

Let $\Omega := \left\{ \lambda \in \mathbb{R}^3_+ \mid (\mathcal{C}_3)^\lambda \supseteq \mathrm{conv}(\mathcal{C}_3) \text{ and } (\mathcal{C}_3)^\lambda \text{ is good} \right\},$

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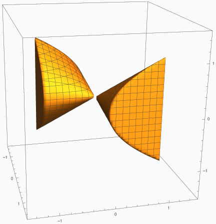
$$\overline{\operatorname{conv}(\mathcal{C}_3)} = \bigcap_{\lambda \in \Omega} (\mathcal{C}_3)^\lambda.$$

Counterexamples

$m = 3$ but not satisfying PDLC condition

$$S := \left\{ (x, y, z) \;\middle|\; \begin{array}{rcl} x^2 & < & 1 \\ y^2 & < & 1 \\ -xy + z^2 & < & 0 \end{array} \right\}$$



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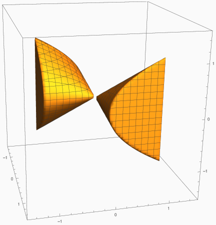

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$$\boxed{\mathrm{conv}(S) \neq \bigcap_{\lambda \in \Omega} S^{\lambda}}$$

m = 4 and satisfying PDLC

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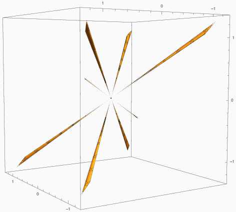

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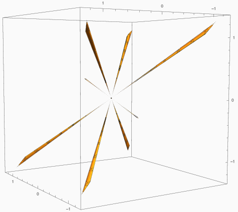

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Do we need a finite number of aggregations?

A non-counterexample:

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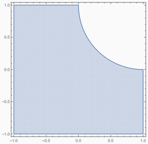

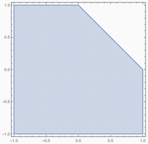

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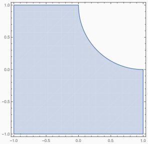

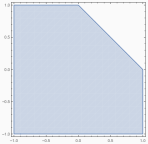

- Let $\Omega^+ := \{\lambda \in \mathbb{R}^3_+ \mid S^\lambda \supseteq \text{conv}(S)\}$
- $\text{conv}(S) = \bigcap_{\lambda \in \Omega^+} S^\lambda$.
- $\text{conv}(S) \subsetneq \bigcap_{\lambda \in \tilde{\Omega}^+} S^\lambda$ for any $\tilde{\Omega}^+ \subseteq \Omega^+$ which is finite.

Do we need a finite number of aggregations?

A non-counterexample:

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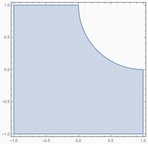

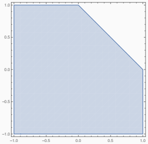

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But PDLC does not hold!

Main proof outline

A new S-Lemma for 3 quadratic constraints

Lemma

Let $n \geq 3$ *and let* $g_1, g_2, g_3 : \mathbb{R}^n \to \mathbb{R}$ *be homogeneous quadratic functions:*

$$g_i(x) = x^\top Q_i x.$$

Assuming there is a linear combination of Q_1, Q_2, Q_3 *that is positive definite, the following equivalence holds*

$$\{x \in \mathbb{R}^n : g_i(x) < 0,\ i \in [3]\} = \emptyset \iff \exists \lambda \in \mathbb{R}^3_+ \setminus \{0\},\ \sum_{i=1}^{3} \lambda_i Q_i \succeq 0.$$

$\mathbf{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda$ proof idea

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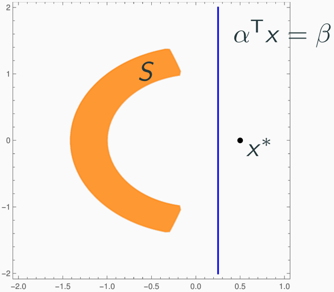
$\text{conv}(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda$:

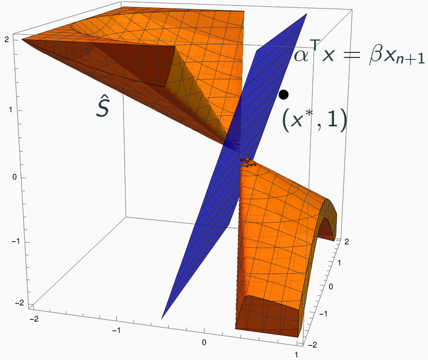
- Pick $x^* \in \mathbb{R}^n$ such that $x^* \notin \text{conv}(S)$. We want to show that is lies outside some aggregation
- Separation theorem $\Rightarrow$ there exists $\alpha^\top x < \beta$ valid for conv(S) that separates x^*.



$\mathbf{conv}(S) = \bigcap_{\lambda \in \Omega} S^{\lambda}$ proof idea

- (Homogenization) The above can be shown to imply: $\{x | \alpha^\top x = \beta x_{n+1}\}$ (call it H) does not intersect homogenization of S:

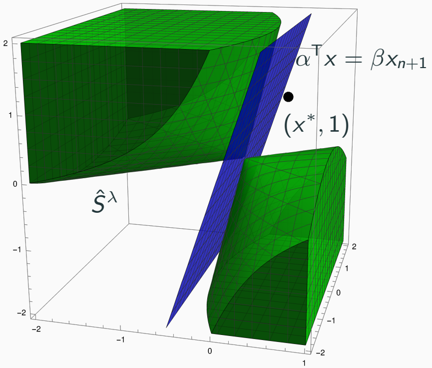
$$H \cap \left\{ (x, x_{n+1}) \,|\, [x \quad x_{n+1}] \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \; i \in [3] \right\} = \emptyset.$$



$\text{conv}(S) = \bigcap_{\lambda \in \Omega} S^{\lambda}$ proof idea

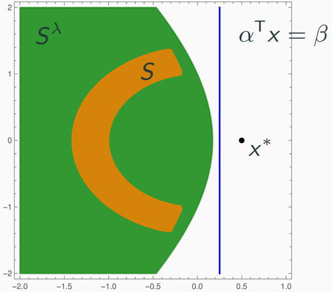
- Applying S-lemma we obtain $\lambda \in \Omega$ such that

$$H \cap \left\{ (x, x_{n+1}) \,|\, [x \quad x_{n+1}] \left(\sum_{i=1}^{3} \lambda_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \right) \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \right\} = \emptyset.$$



$\mathrm{conv}(S) = \bigcap_{\lambda \in \Omega} S^{\lambda}$ proof idea

- Dehomogenizing, we obtain $S^{\lambda} \supseteq \mathrm{conv}(S)$ that excludes x^*



Summary and open questions

- We have shown that, under technical assumptions, aggregations are enough to describe the convex hull of 3 quadratics
- We have also shown that the result is not true if some conditions are relaxed.

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- We do not completely understand the PDLC condition. What is its geometrical meaning? Can we replace it by another condition and obtain a similar result?

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Thank you!