The Complexity of Pacing for Second-Price Auctions

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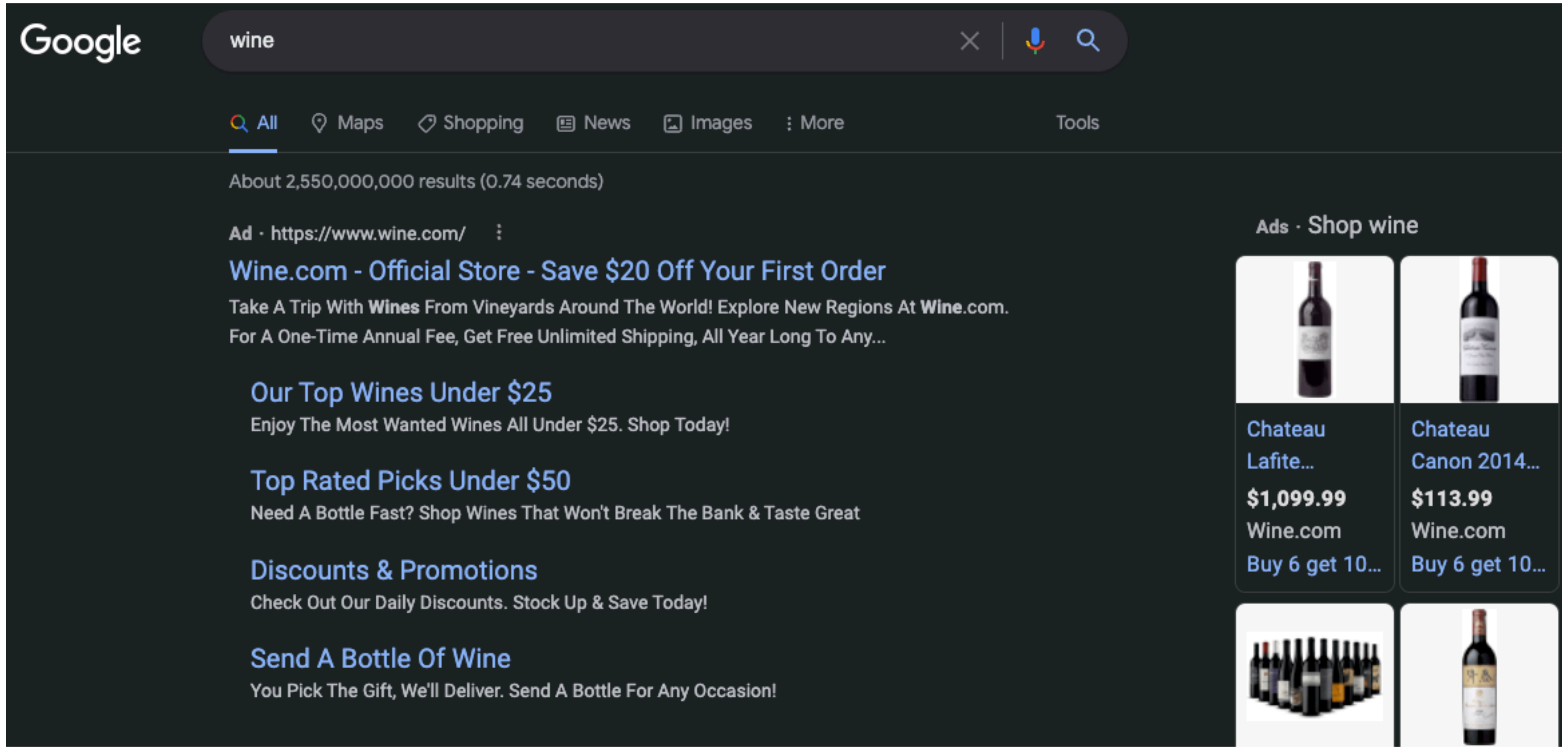
Joint work with Xi Chen (Columbia) and Rachitesh Kumar (Columbia)

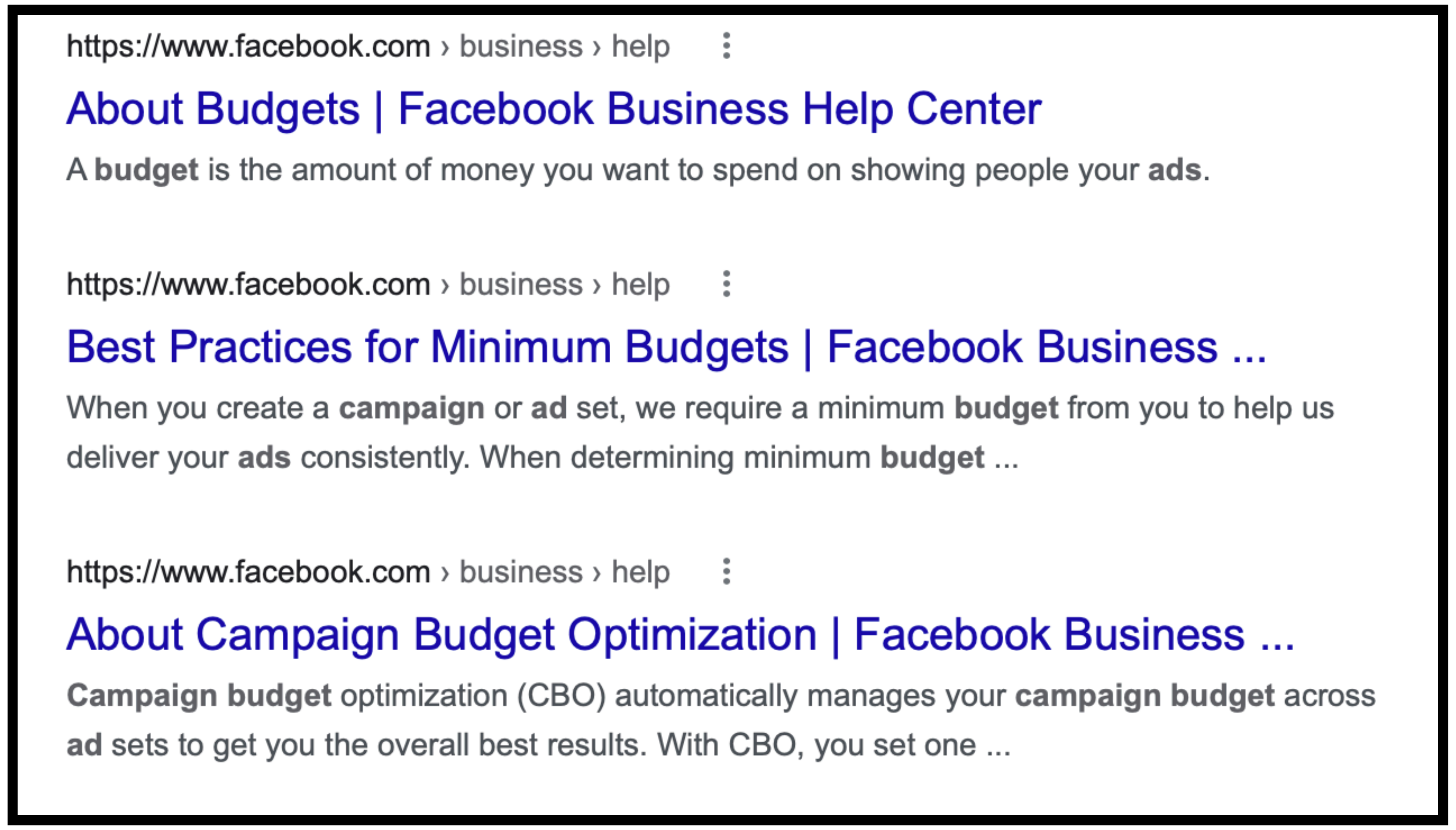
The Face of Professor
Dan the Man



Motivation

Online Ad Auctions

- The second-price auction or its generalizations are often used to sell ad slots
- Advertisers are often budget constrained, so they can't bid their value in all available auctions
- Platforms often manage budgets on behalf of advertisers


https://www.facebook.com › business › help

About Budgets | Facebook Business Help Center

A **budget** is the amount of money you want to spend on showing people your **ads**.

https://www.facebook.com › business › help

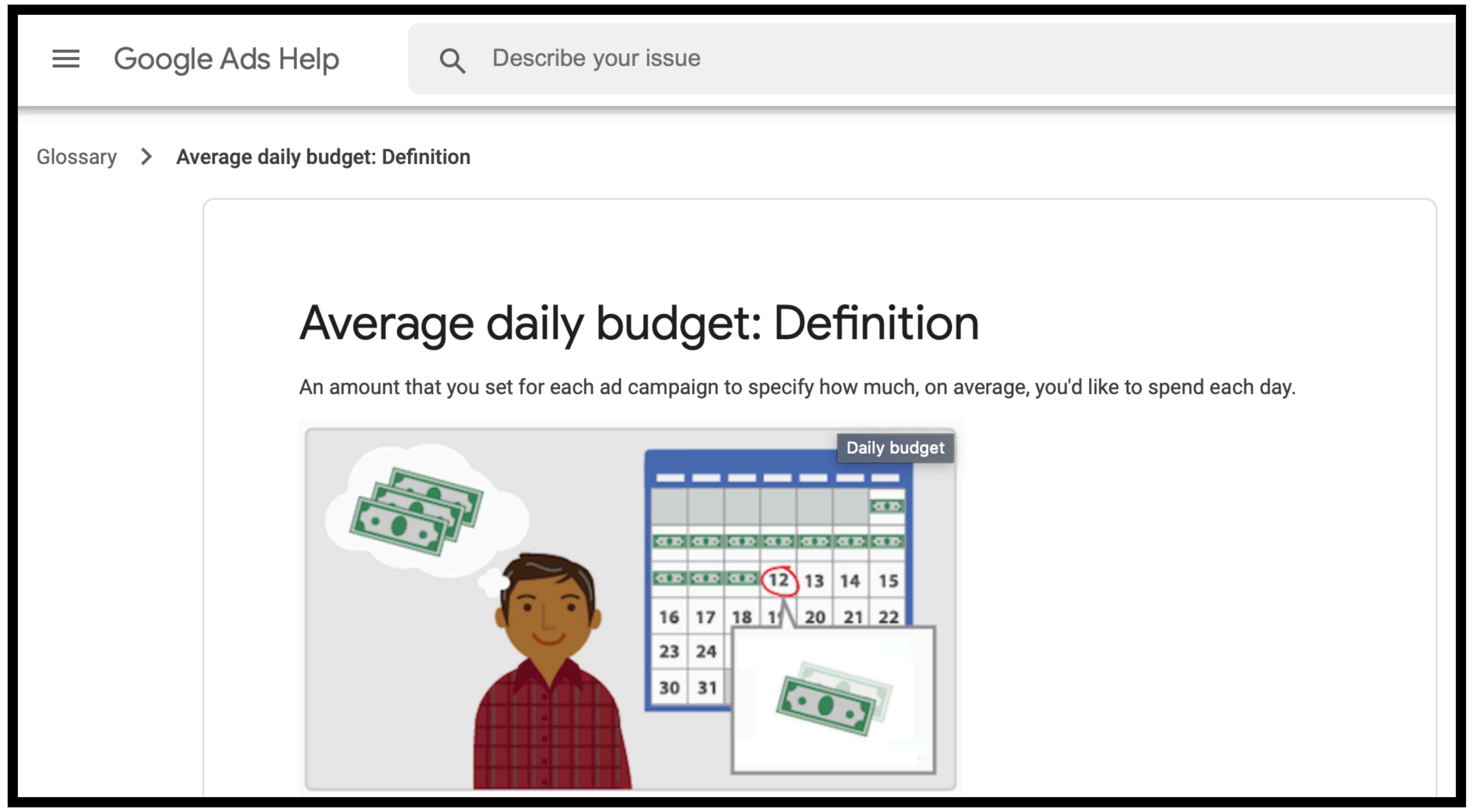
Best Practices for Minimum Budgets | Facebook Business ...

When you create a **campaign** or **ad** set, we require a minimum **budget** from you to help us deliver your **ads** consistently. When determining minimum **budget** ...

https://www.facebook.com › business › help

About Campaign Budget Optimization | Facebook Business ...

Campaign budget optimization (CBO) automatically manages your **campaign budget** across **ad** sets to get you the overall best results. With CBO, you set one ...


Google Ads Help

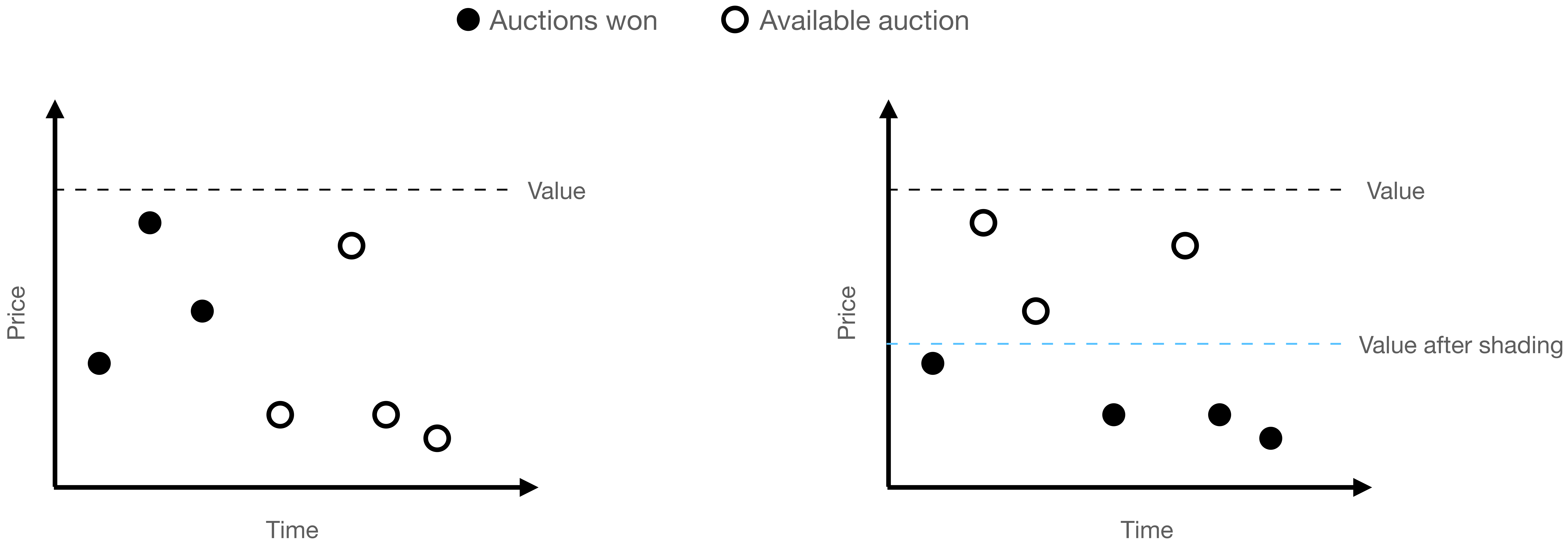
Describe your issue

Glossary > Average daily budget: Definition

Average daily budget: Definition

An amount that you set for each ad campaign to specify how much, on average, you'd like to spend each day.

Pacing



Naive: Participate until you run out of budget

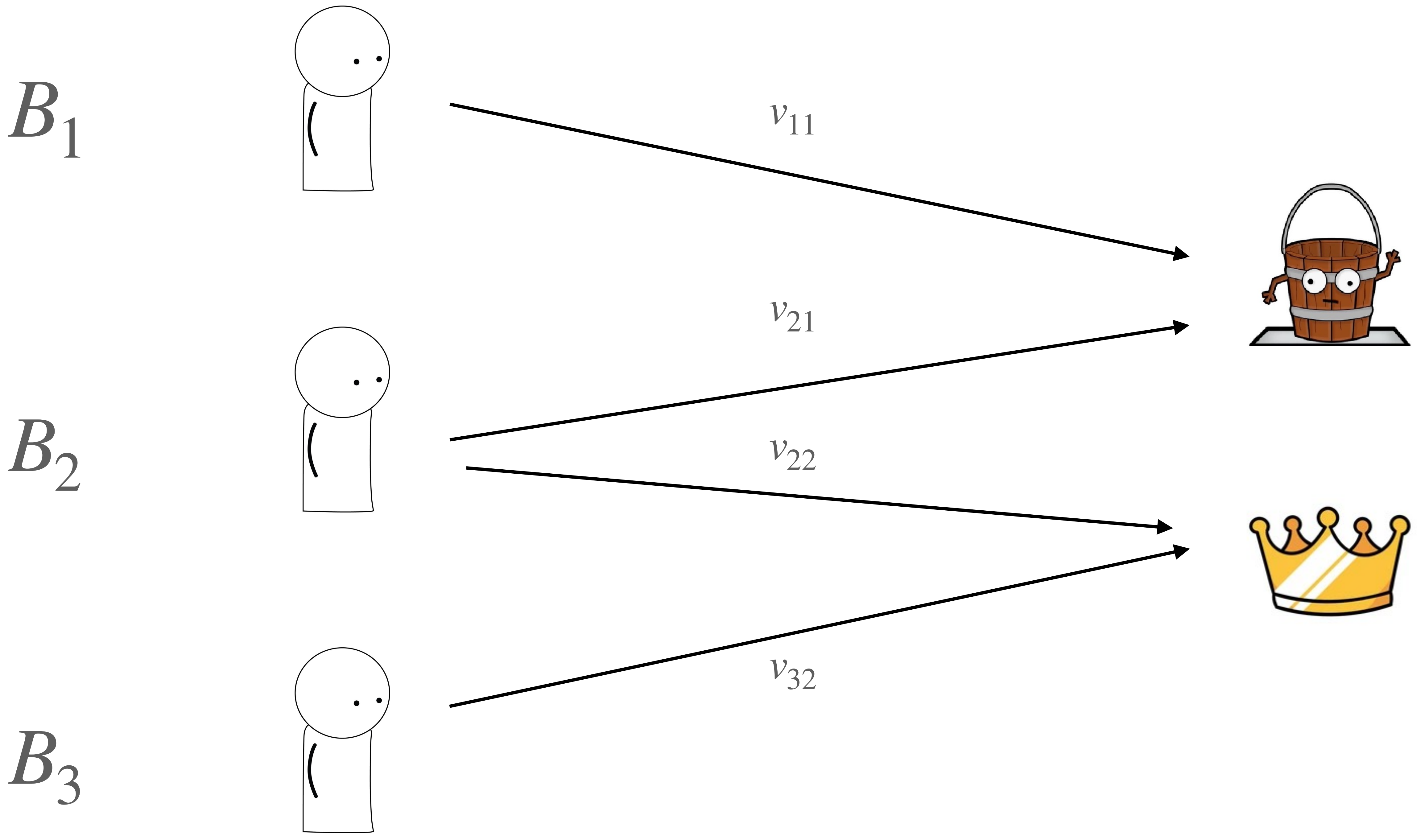
Pacing: Bid by multiplicatively shading value to spend budget more evenly

Model

Buyer i has value v_{ij} for good j

Budget B_i

n Buyers



m Goods

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Pacing Multiplier

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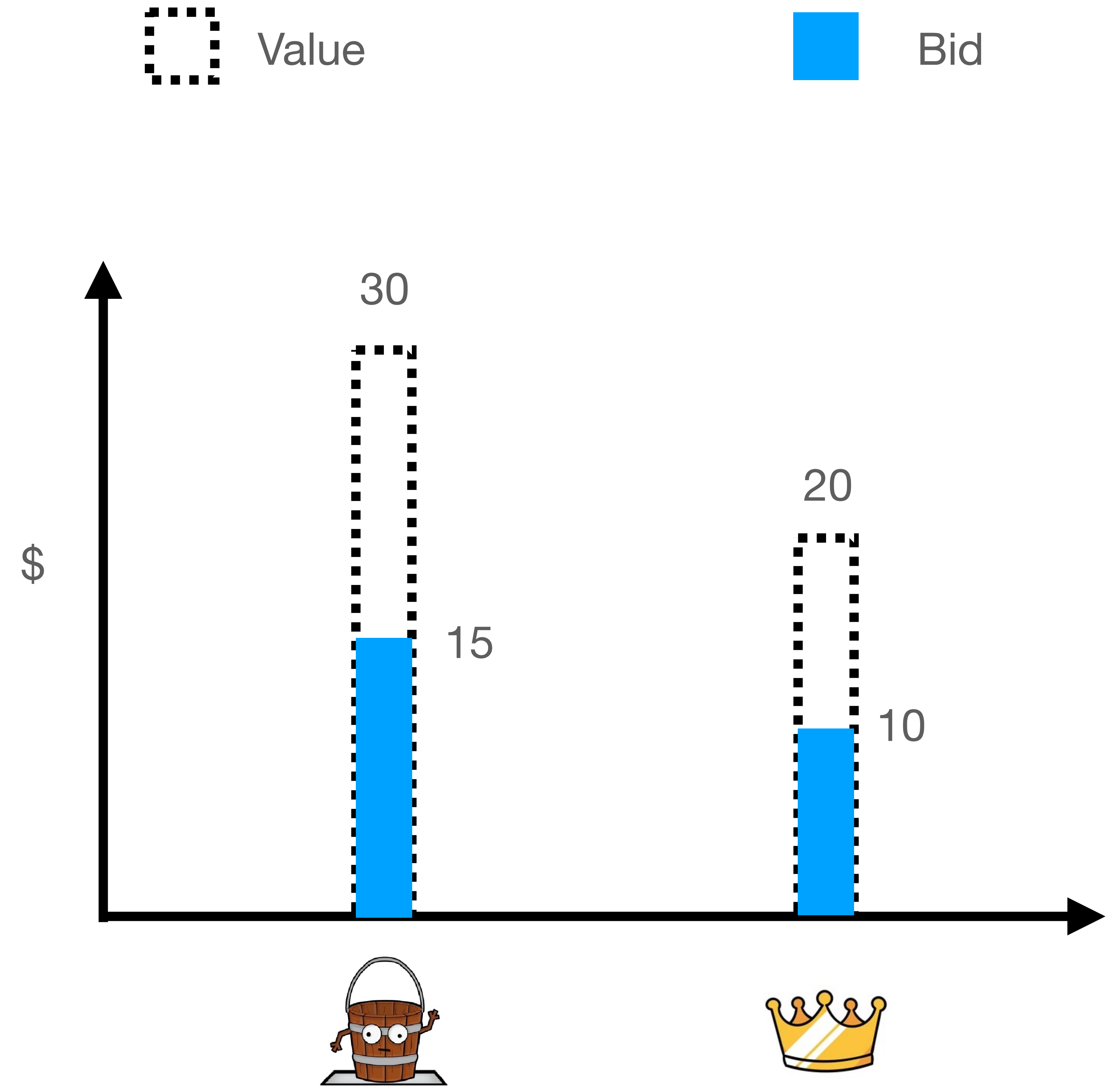


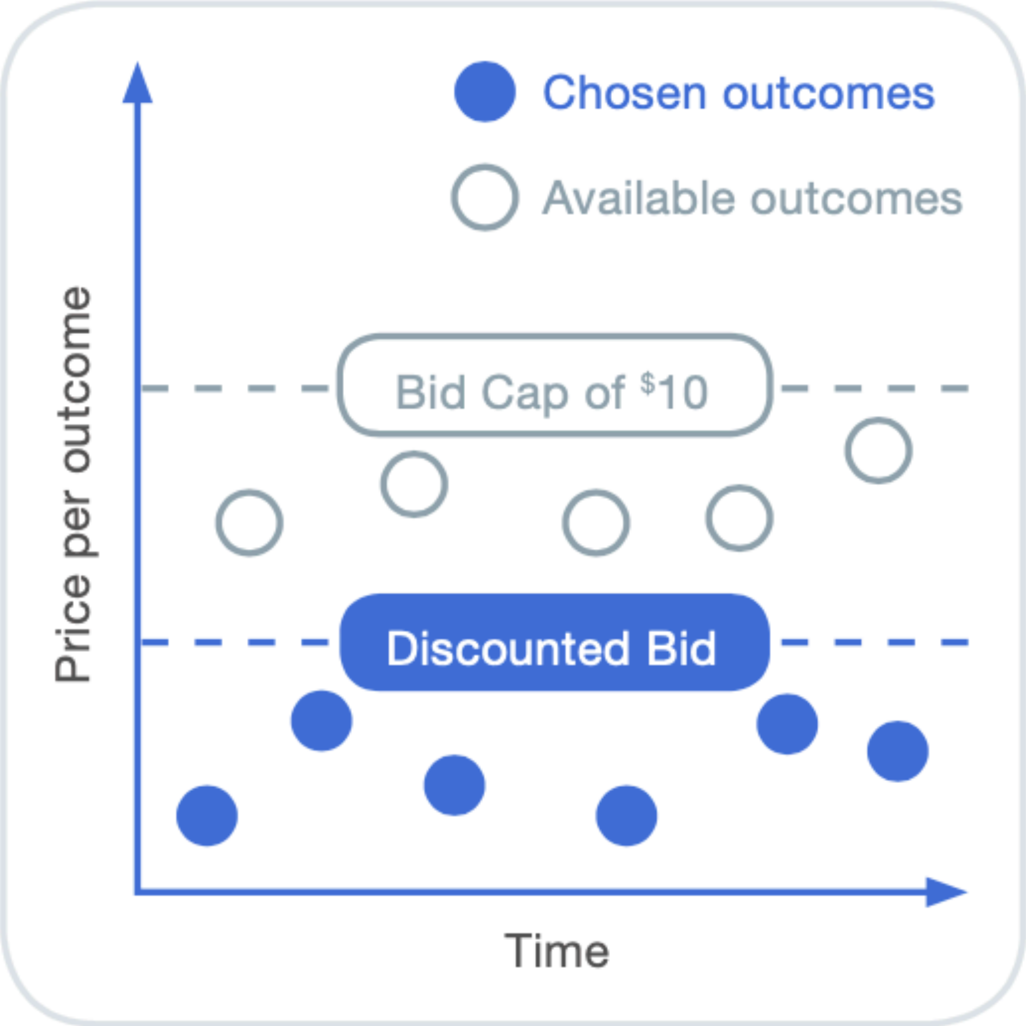
$$\alpha = 1/2$$

Why Pacing?: Used in Practice

Lowest cost bid strategy with standard delivery enabled

This uses "discount" pacing to spend budget on results with the lowest costs and aims to spend budget evenly over the course of your campaign. This option maximizes advertiser value by minimizing your cost per result.

> **Example:** Assume you set a bid cap of $10. As your ad enters each auction, we may "discount" your bid to ensure we can spend your full budget over the duration of the ad set. The mechanism of lowering or "discounting" your bid means that you might not win every auction you could have, but you'll have a chance to capture more outcomes at more efficient costs over the course of your campaign, instead of using up your budget too quickly on more expensive results. Note that the discounted bid for a particular auction might be higher or lower based on how much budget has been spent and the time elapsed in the campaign.



Source: Facebook Guide for Advertisers

Why Pacing?: Theoretically Optimal

Related Works with Pacing-Based Optimal Strategies

- **Gummadi, Key, Proutiere (2012):** Pacing is optimal in MDP-based repeated auction model for budget-constrained bidders
- **Balseiro, Besbes, Weintraub (2015):** Pacing-based bidding strategies form a Fluid Mean Field Equilibrium for buyers with budget constraints, stochastic values
- **Balseiro and Gur (2019):** Pacing is optimal under stochastic and adverserial inputs for budget-constrained buyer participating in repeated auctions

Central Questions

What happens when every buyer uses pacing simultaneously?

Do pacing based strategies efficiently converge to equilibrium for repeated auctions?

Pacing Game

Conitzer, Kroer, Sodomka, Stier-Moses (WINE'18, OR'22)

- Buyer i has strategy $\alpha_i \in [0,1]$

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- $h_j(\overrightarrow{\alpha})$: Highest paced bid, i.e., largest element in $\alpha_1 v_{1j}, \ldots, \alpha_n v_{nj}$

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- $h_j(\overrightarrow{\alpha})$: Highest paced bid, i.e., largest element in $\alpha_1 v_{1j}, \ldots, \alpha_n v_{nj}$
- $p_j(\overrightarrow{\alpha})$: Second-highest paced bid, i.e., second-largest element in $\alpha_1 v_{1j}, \ldots, \alpha_n v_{nj}$

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3. Budget constraint: $\sum_j x_{ij} p_j(\vec{\alpha}) \leq B_i$
4. **No unnecessary pacing:** $\sum_j x_{ij} p_j(\vec{\alpha}) < B_i$ implies $\alpha_i = 1$

Approximate Pacing Equilibrium

- A (δ, γ)-approximate pacing equilibrium is a tuple $(\vec{\alpha}, \vec{x})$ such that:
  1. Close to highest bid wins: $x_{ij} > 0$ implies $\alpha_i v_{ij} \geq (1 - \delta)\ h_j(\vec{\alpha})$
  2. Full allocation: $h_j(\vec{\alpha}) > 0$ implies $\sum_i x_{ij} = 1$
  3. Budget constraint: $\sum_j x_{ij} p_j(\vec{\alpha}) \leq B_i$
  4. Not too much unnecessary pacing: $\sum_j x_{ij} p_j(\vec{\alpha}) < (1 - \gamma)\ B_i$ implies $\alpha_i \geq 1 - \gamma$

Our Results

- **Theorem:** For any constant $c > 0$, computing a (δ, γ)-approximate pacing equilibrium is PPAD-hard for $\delta = \gamma = 1/n^c$.

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Our Results

- **Theorem:** For any constant $c > 0$, computing a (δ, γ)-approximate pacing equilibrium is PPAD-hard for $\delta = \gamma = 1/n^c$.
- **Theorem:** Computing an (exact) pacing equilibrium is in PPAD.
- In simple terms, computing pacing equilibria is as hard as computing Nash equilibria of bimatrix games or finding a Brouwer fixed point, which have eluded efficient algorithms.

Implications for Related Work

Assuming PPAD-hard problems are not efficiently solvable

- **Conitzer, Kroer, Sodomka, Stier-Moses (WINE'18, OR'22):** Show existence of pacing equilibria via a limit argument, show relationship to Nash Eq., conjecture that computing it is PPAD-complete

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- **Borgs, Chayes, Immorlica, Jain, Etesami, Mahdian (WWW'07):** Give tatonnement-style dynamics for pacing, show efficient convergence for first-price auctions and conjecture similar convergence for second-price auctions

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  ➡ Dynamics cannot converge efficiently for second-price auctions

Proof Idea: PPAD Hardness

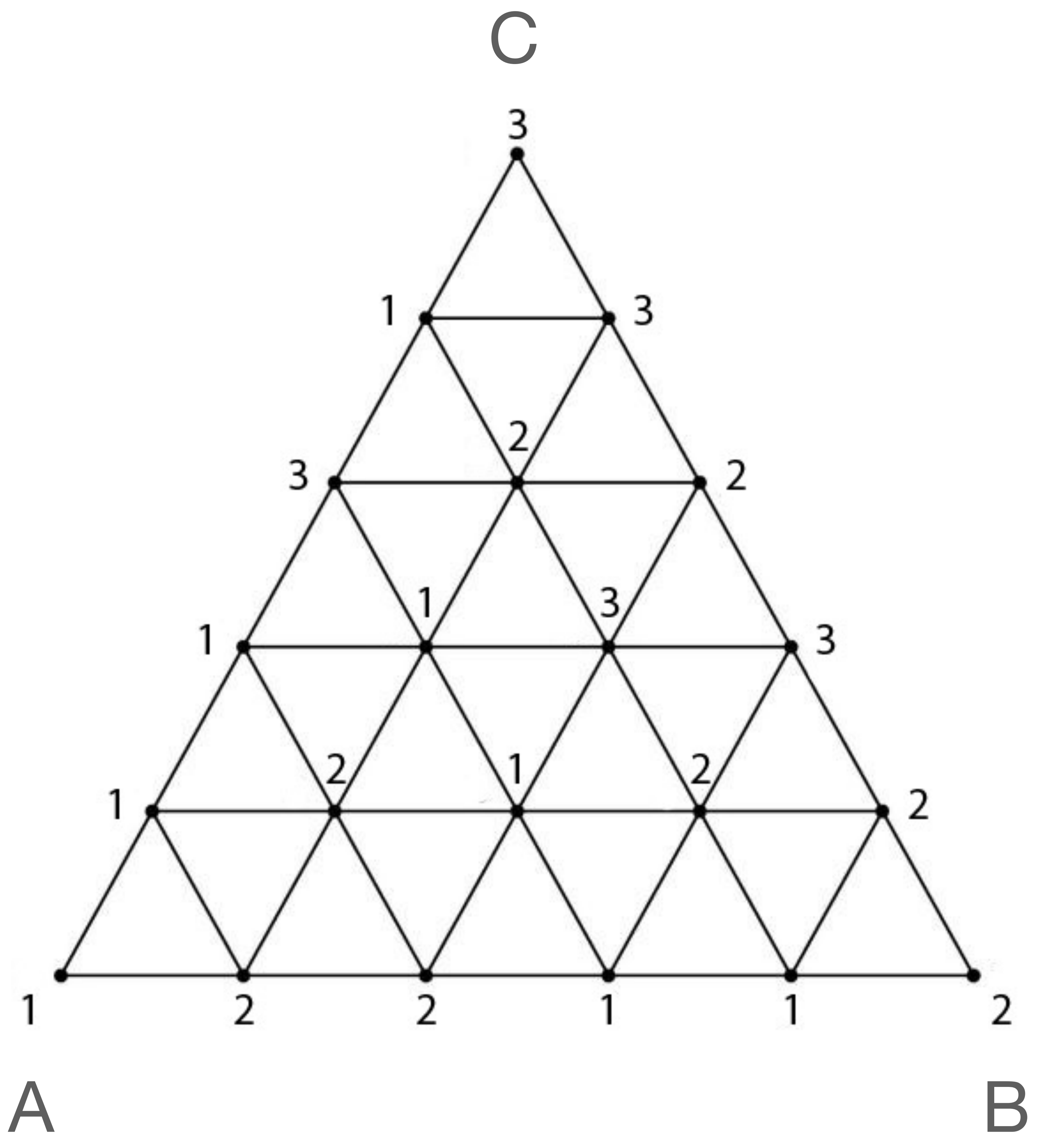
- Reduce from the problem of computing Nash equilibria in two player 0-1 cost bimatrix games, which is known to be PPAD-complete
- Has the important implications discussed earlier, but proof has little intuition

Proof Idea:

Computing approximate pacing equilibrium is in PPAD

- Sperner's Lemma: Given a triangle ABC which has been triangulated into smaller triangles, if
  - A, B and C are labelled 1, 2 and 3 respectively
  - Each vertex on an edge of ABC is labelled only with one of the two labels of the ends of its edge

  Then, there is a triangle which has all three labels



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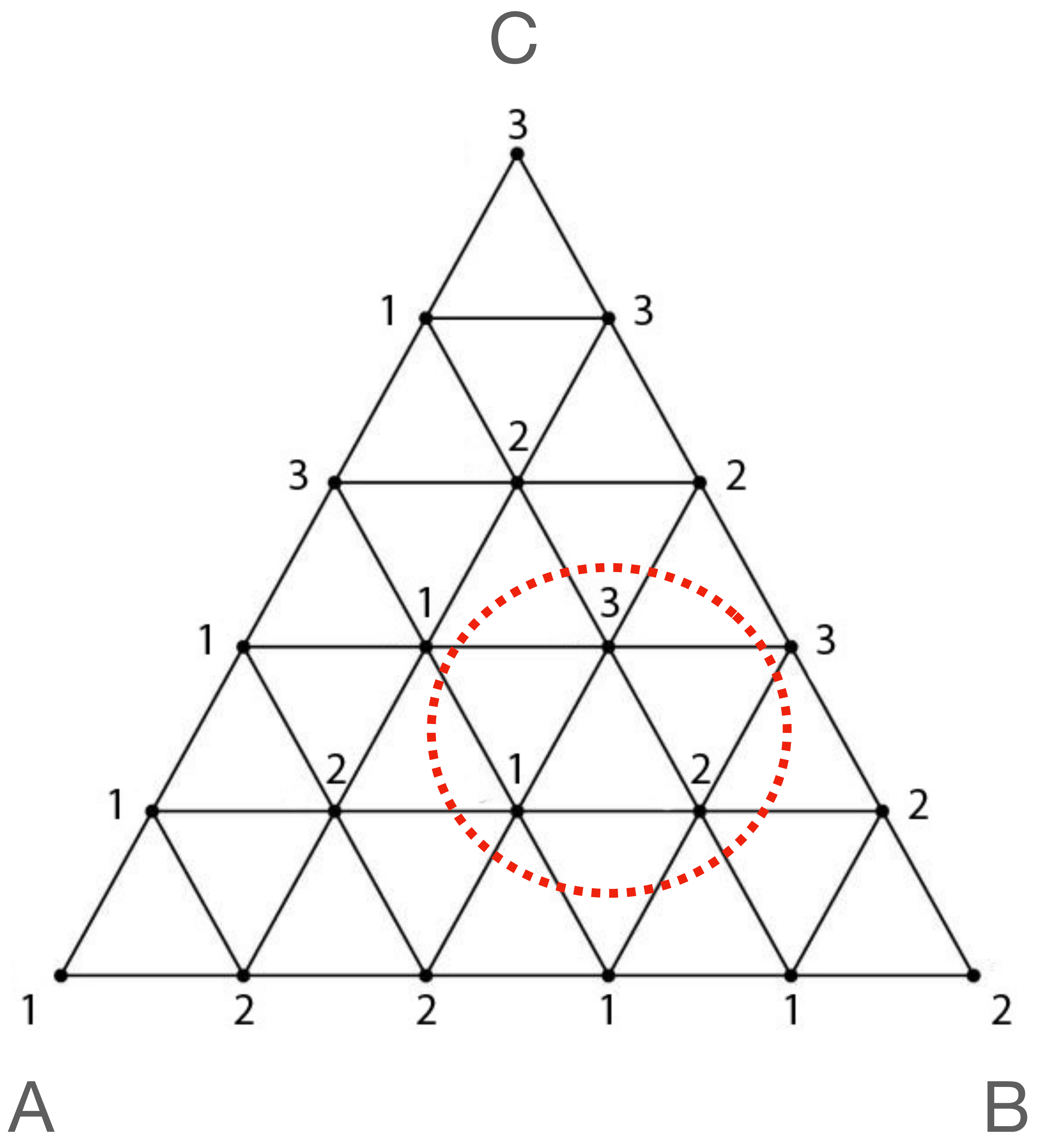


Proof Idea

Computing approximate pacing equilibrium is in PPAD

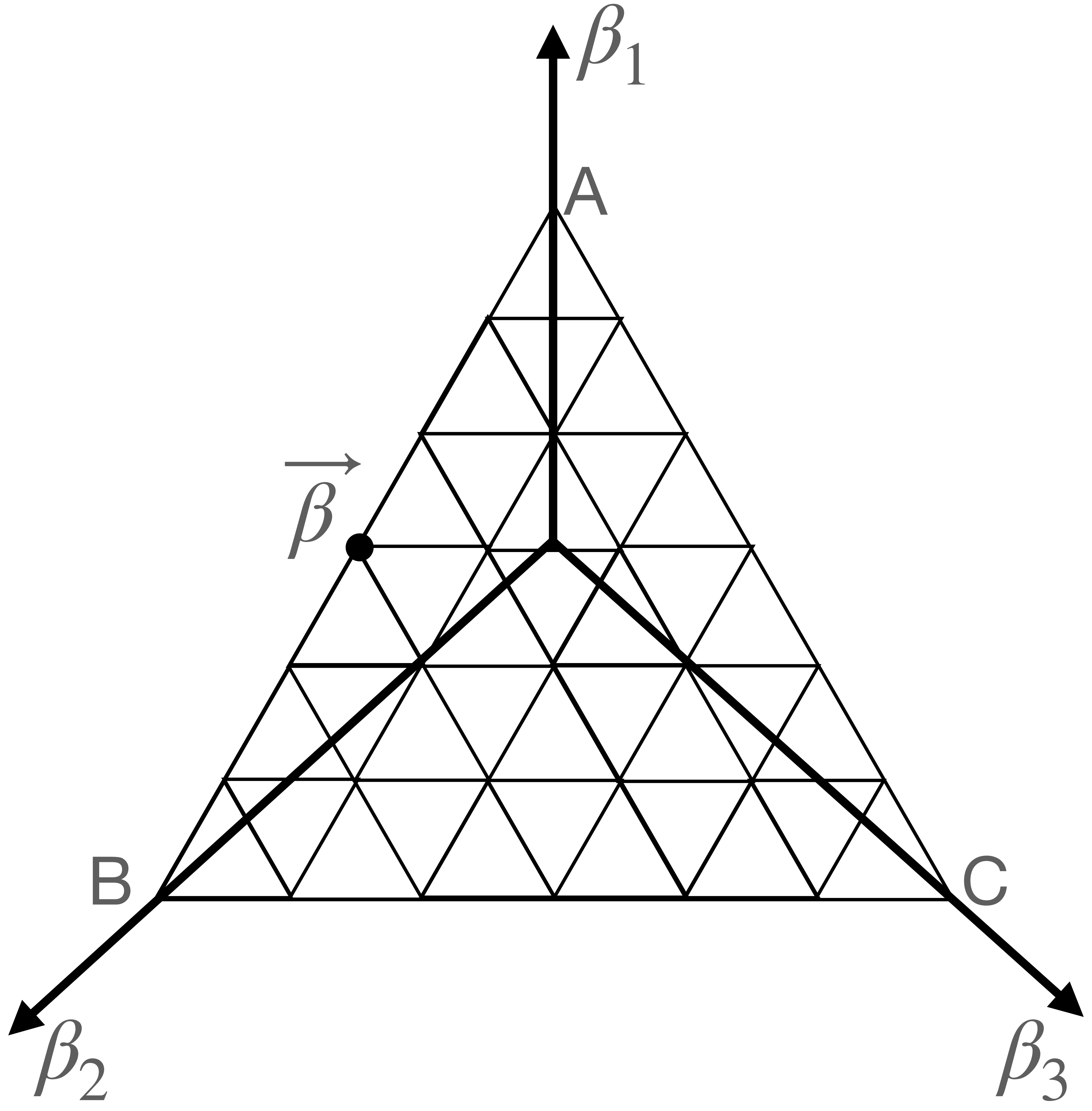
- **Smooth tie-breaking rule:** Allocation rule is a continuous function of pacing multipliers and only allows “close to highest” bids to win.

$$x_{ij}(\alpha) := \frac{[\alpha_i v_{ij} - (1-\delta)h_j(\alpha)]^+}{\sum_{r\in[n]} [\alpha_r v_{rj} - (1-\delta)h_j(\alpha)]^+}$$

Proof Idea

Computing approximate pacing equilibrium is in PPAD

- Let ABC = $\{ \overrightarrow{\beta} \in \mathbb{R}^3_+ \mid \beta_1 + \beta_2 + \beta_3 = 1 \}$
- Label a vertex $\overrightarrow{\beta}$ in ABC as follows:
  - Set $\alpha_i = t \cdot \beta_i$
  - Increase t gradually and instruct each buyer to say “Stop” if her budget constraint becomes tight or her pacing multiplier reaches 1
  - Label $\overrightarrow{\beta}$ with $k \in \{1,2,3\}$ if buyer k is the first one to say “Stop”



Proof Idea: PPAD Membership of Exact Eq.

- Use Sperner's Lemma to show PPAD membership of (δ, γ)-approximate pacing equilibria
- Give rounding algorithm to get PPAD membership of computing $(0,\gamma)$-approximate pacing equilibrium (only highest bidder wins, some unnecessary pacing is allowed)
- LP-based argument to get PPAD membership of computing exact pacing equilibrium

MIP Approach to SPPE [CKSS'18/'22]

- $\alpha_i \in [0,1]$: Buyer i's pacing multiplier
- $s_{ij} \in \mathbb{R}_+$: Buyer i's spend on good j
- $p_j \in \mathbb{R}_+$: Price of good j
- $h_j \in \mathbb{R}_+$: The highest bid for good j

- $d_{ij} \in \{0,1\}$: 1 if buyer i may win any part of good j
- $y_i \in \{0,1\}$: 1 if buyer i spends its full budget
- $w_{ij} \in \{0,1\}$: 1 if buyer i is the winner of good j
- $r_{ij} \in \{0,1\}$: 1 if buyer i is the second price for good j

Conitzer, Kroer, Sodomka, Stier, Multiplicative Pacing Equilibria in Auction Markets. OR'22

MIP Approach to SPPE

$$\sum_{j \in M} s_{ij} \leq B_i \quad (\forall i \in N) \tag{1}$$

$$\sum_{j \in M} s_{ij} \geq y_i B_i \quad (\forall i \in N) \tag{2}$$

$$\alpha_i \geq 1 - y_i \quad (\forall i \in N) \tag{3}$$

$$\sum_{i \in N} s_{ij} = p_j \quad (\forall j \in M) \tag{4}$$

$$s_{ij} \leq B_i d_{ij} \quad (\forall i \in N, j \in M) \tag{5}$$

$$h_j \geq \alpha_i v_{ij} \quad (\forall i \in N, j \in M) \tag{6}$$

$$h_j \leq \alpha_i v_{ij} + (1 - d_{ij})\bar{v}_j \quad (\forall i \in N, j \in M) \tag{7}$$

$$w_{ij} \leq d_{ij} \quad (\forall i \in N, j \in M) \tag{8}$$

$$p_j \geq \alpha_i v_{ij} - w_{ij} v_{ij} \quad (\forall i \in N, j \in M) \tag{9}$$

$$p_j \leq \alpha_i v_{ij} + (1 - r_{ij})\bar{v}_j \quad (\forall i \in N, j \in M) \tag{10}$$

$$\sum_{i \in N} w_{ij} = 1 \quad (\forall j \in M) \tag{11}$$

$$\sum_{i \in N} r_{ij} = 1 \quad (\forall j \in M) \tag{12}$$

$$r_{ij} + w_{ij} \leq 1 \quad (\forall i \in N, j \in M) \tag{13}$$

Conitzer, Kroer, Sodomka, Stier, Multiplicative Pacing Equilibria in Auction Markets. OR'22

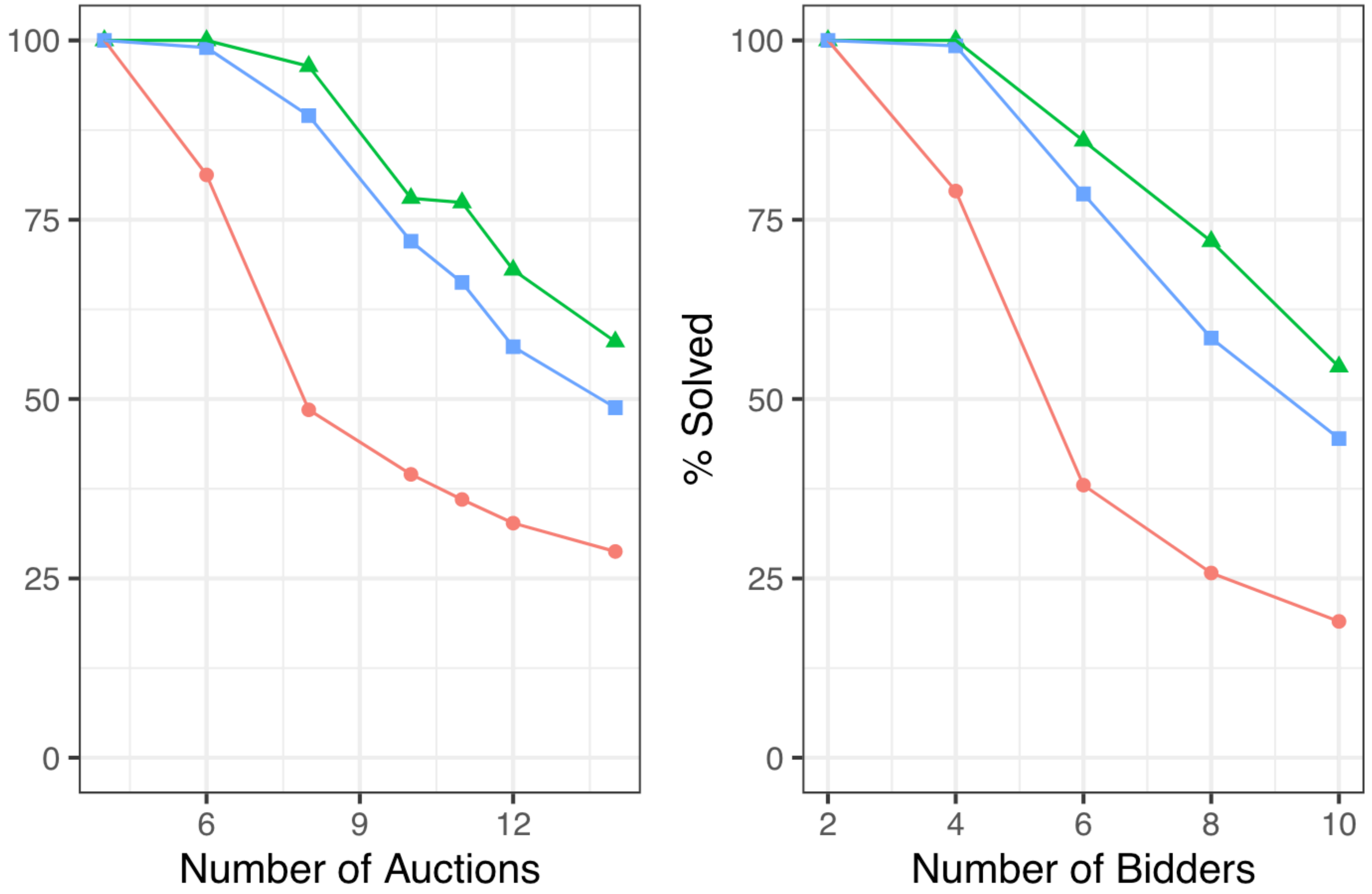
MIP Approach to SPPE

- So does it work?

Conitzer, Kroer, Sodomka, Stier, Multiplicative Pacing Equilibria in Auction Markets. OR'22

MIP Approach to SPPE

- So does it work?
- No :(



Conitzer, Kroer, Sodomka, Stier, Multiplicative Pacing Equilibria in Auction Markets. OR'22

Conclusion

- Introduced second-price pacing equilibrium (SPPE)
- We show that computing an SPPE is a PPAD-complete problem
- Resolved several open problems in budget management literature
- Open problems:
  - Better MIP approach to computing SPPE?
  - Approximation algorithms?
  - Complementarity-based algorithms?

Thanks!

- Christian Kroer, Assistant Professor, IEOR Dept, Columbia University
- Get in touch:
  - christian.kroer@columbia.edu
  - www.christiankroer.com
- Talk based on:
  - The Complexity of Pacing for Second-Price Auctions. Xi Chen, Kroer, Rachitesh Kumar. ACM EC'21
  - Multiplicative Pacing Equilibria in Auction Markets. Vincent Conitzer, Kroer, Eric Sodomka, Nico Stier. WINE'18, OR'22
- Find the papers on my website or arXiv