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Extending Generalized Disjunctive Programming to Model Hierarchical Systems

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2022 MIXED INTEGER PROGRAMMING WORKSHOP feat. DANniversary

Mixed Integer Programming Workshop 2022
DIMACS, Rutgers University
May 23-26, 2022

Congratulations Dan!

Generalized Disjunctive Programming *(GDP)*

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Raman and Grossmann (1994) *(Extension Balas, 1979)*

$$\min\ Z = \sum_k c_k + f(x)$$ Objective Function

$$s.t.\quad r(x) \le 0$$ Common Constraints

OR operator ⟶

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \le 0 \\ c_k = \gamma_{jk} \end{bmatrix}, k \in K$$

Disjunction

Constraints

Fixed Charges

$$\Omega(Y) = true$$ Logic Propositions

$$x \in R^n, c_k \in R^1$$ Continuous Variables

$$Y_{jk} \in \{true, false\}$$ Boolean Variables

a) Provides a *"high level"* modeling representation

b) Can be used to derive MI(N)LP models *(algebraic constraints)*

Goal: introduce extension to Nested GDP

GDP is a higher level of representation for MILP/MINLP

Optimization problem with algebraic expressions, disjunctions & logic propositions

Convex GDP

$$\min z = f(x)$$ *Objective Function*

Convex

$$s.t.\ g(x) \leq 0$$ *Global Constraints*

$$\vee_{i \in D_k} \begin{bmatrix} Y_{ki} \\ r_{ki}(x) \leq 0 \end{bmatrix} \quad k \in K$$

$$\underline{\vee}_{i \in D_k} Y_{ki} \qquad k \in K$$

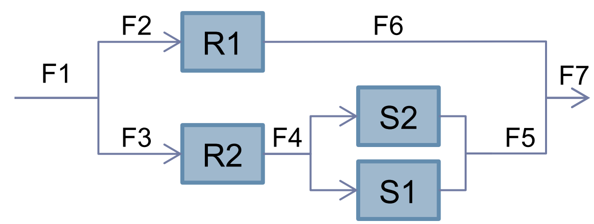
Disjunctions

$$\Omega(Y) = True$$ *Logic Propositions*

$$x \in \mathbb{R}^n$$

$$Y_{ki} \in \{True, False\} \qquad k \in K, i \in D_k$$

Illustration – Process network



$$\max z = P_7 F_7 - P_1 F_1 - c_R - c_S$$ *Objective function*

$$F_1 = F_2 + F_3$$
$$F_7 = F_5 + F_6$$ *Global Constraints*

$$\begin{bmatrix} Y_{R1} \\ F_6 = \beta_{R1} F_2 \\ F_3 = F_4 = 0 \\ c_R = \gamma_{r1} \end{bmatrix} \vee \begin{bmatrix} Y_{R2} \\ F_6 = F_2 = 0 \\ F_4 = \beta_{R2} F_3 \\ c_R = \gamma_{r2} \end{bmatrix}$$

$$\begin{bmatrix} Y_{S1} \\ F_5 = \beta_{S1} F_4 \\ c_S = \gamma_{s2} \end{bmatrix} \vee \begin{bmatrix} Y_{S2} \\ F_5 = \beta_{S2} F_4 \\ c_S = \gamma_{s2} \end{bmatrix} \vee \begin{bmatrix} Y_{S\_NO} \\ F_5 = 0 \\ c_S = 0 \end{bmatrix}$$ *Disjunctions*

$$Y_{R1} \underline{\vee} Y_{R2}$$
$$Y_{S1} \underline{\vee} Y_{S2} \underline{\vee} Y_{S\_NO}$$

$$Y_{R1} \Leftrightarrow Y_{S\_NO}$$ *Logic*

1. Raman R. and Grossmann I.E., "Modelling and Computational Techniques for Logic-Based Integer Programming", Computers and Chemical Engineering, 18, 563, 1994.

Methods Generalized Disjunctive Programming

GDP

- **Logic based methods**
  - **Branch and bound**
    - *(Lee & Grossmann, 2000)*
    - *(Bernal & Grossmann, 2020)*
  - **Decomposition**
    - *Outer-Approximation*
    - *Generalized Benders*
    - *(Turkay & Grossmann, 1997)*
    - *(Chen and Grossmann, 2019)*
- **Reformulation MINLP**
  - **Hull relaxation**
  - **Big-M**
  - *Branch and Bound*
  - *Outer-Approximation*
  - *Generalized Benders*
  - *Extended Cutting Plane*
  - *MILP for linear functions*

Big-M MINLP (BM)

- **MINLP reformulation of GDP**

$$\min Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)$$

$$s.t. \quad r(x) \le 0$$

$$g_{jk}(x) \le M_{jk}(1-\lambda_{jk}) \quad ,j \in J_k, k \in K$$

Big-M Parameter

$$\sum_{j \in J_k} \lambda_{jk} = 1, \quad k \in K$$

$$A\lambda \le a$$

Logic constraints

Williams (1990)

$$x \ge 0, \lambda_{jk} \in \{0,1\}$$

NLP Relaxation $0 \le \lambda_{jk} \le 1$ **=> Lower bound to optimum of GDP**

Hull Relaxation Problem (HRP)

(Lee, Grossmann, 2000)

HRP:

$$\min Z = \sum_{k\in K}\sum_{j\in J_k}\gamma_{jk}\lambda_{jk} + f(x)$$

$$s.t. \quad r(x) \le 0$$

$$x = \sum_{j\in J_k} \nu^{jk}, k \in K$$

$$0 \le \nu^{jk} \le \lambda_{jk} U_{jk}, j \in J_k, k \in K$$

$$\sum_{j\in J_k} \lambda_{jk} = 1, k \in K$$

$$\lambda_{jk} g_{jk}(\nu^{jk}/\lambda_{jk}) \le 0, \quad j \in J_k, k \in K$$

$$A\lambda \le a$$

$$x, \nu^{jk} \ge 0, \; 0 \le \lambda_{jk} \le 1, \; j \in J_k, k \in K$$

Disaggregated variables

Convex Hull each disjunction

Perspective function

Logic constraints

- **Property**: *The NLP (HRP) yields a lower bound to optimum of (GDP).*

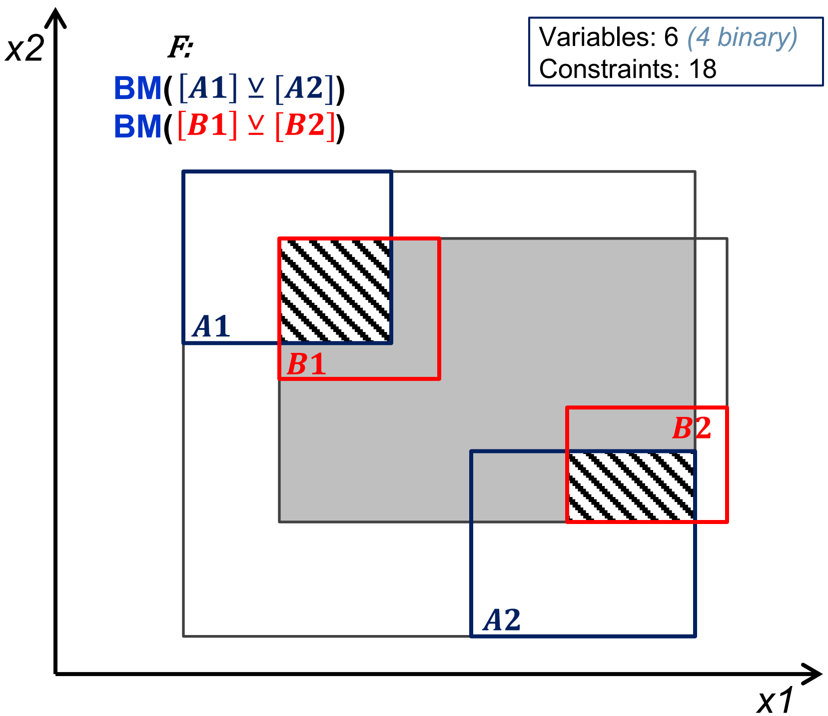
MINLP reformulation: set $\lambda_{jk} = 0,1$

Hull relaxation: intersection of convex hull of each disjunction

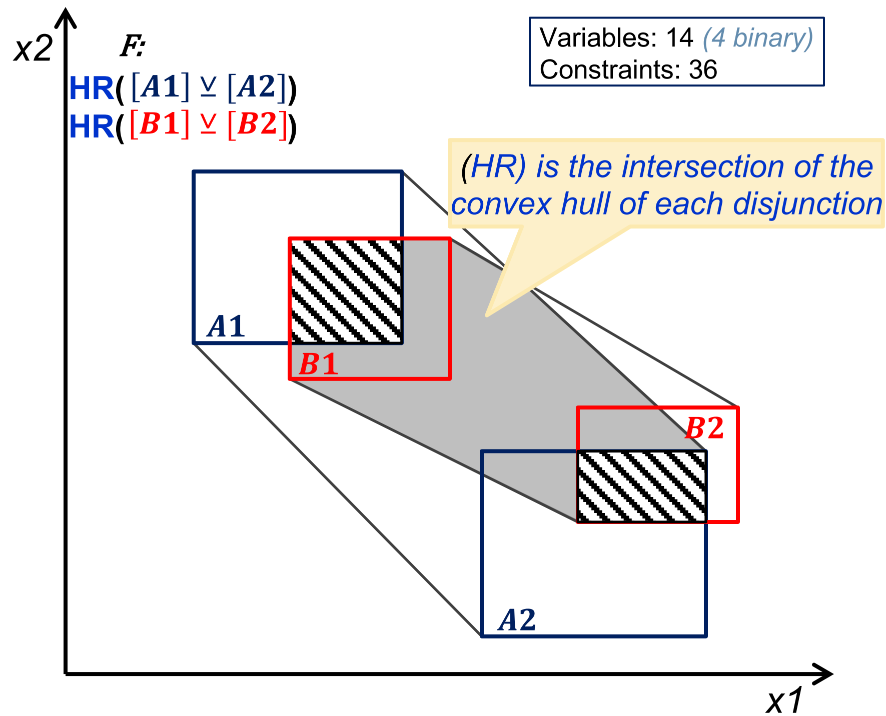
(HR) provides a tighter relaxation than (BM)

Illustration of (BM) and (HR) relaxations

(BM)



(HR)

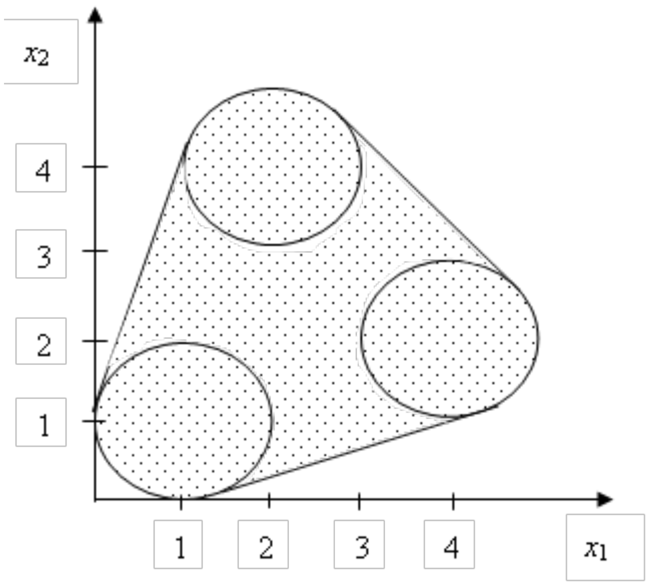


Tradeoff: (BM) has a smaller problem size while (HR) has a tighter continuous relaxation

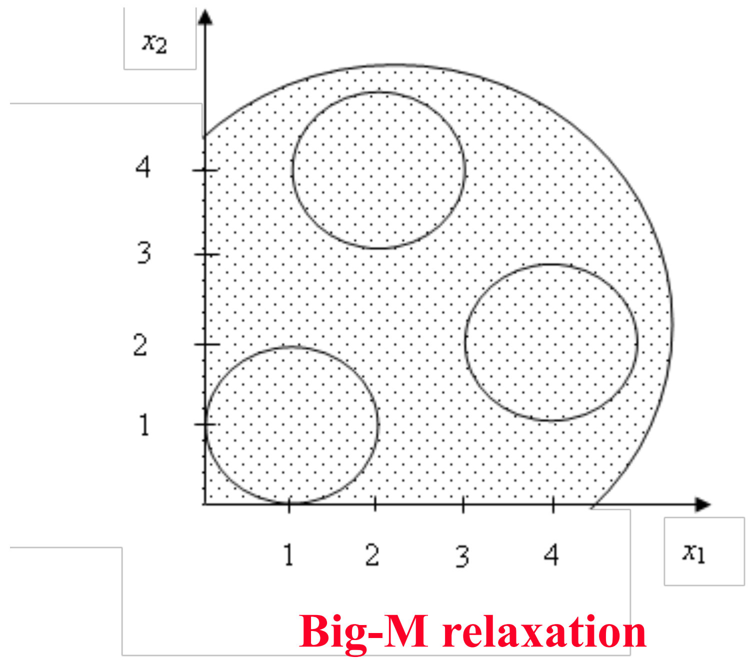
Strength Lower Bounds

- **Theorem**: *The relaxation of (HRP) yields a lower bound that is greater than or equal to the lower bound that is obtained from the relaxation of problem (BM*

Grossmann, Lee (2003)



Convex hull relaxation



Big-M relaxation

Convex Hull of a set of disjunctions is smallest convex set that includes set of disjunctions.

Projected relaxation of (CH) onto the space of (BM) is as tight or tighter than that of (BM)

Nested GDP (NGDP): Hierarchical Logic

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

$$\bigvee_{j \in J_i} \begin{bmatrix} Y_{ij} \\ g_{ij}(x) \le 0 \\ \\ \end{bmatrix} \quad \forall i \in I$$

$$\Omega(Y \quad) = True$$

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \quad \forall i \in I, j \in J_i$$

For problems with a hierarchical structure, there are lower-level decisions (*W*) that are subject to upper-level decisions (*Y*).

Examples:

- ***Process design (upper-level) and process operation (lower-level)***
- ***Long-term planning (upper-level) and short-term scheduling (lower-level)***

NGDP Formulations

Traditional GDP does not consider nested disjunctions and requires transforming model into an Equivalent Single-level GDP (*Approach 1*):

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

Single-Level Disjunctions

$$\bigvee_{j \in J_i} \begin{bmatrix} Y_{ij} \\ g_{ij}(x) \le 0 \end{bmatrix} \quad \forall i \in I$$

Extract nested disjunction (requires creating a new Boolean variable W_{ijk0})

$$\bigvee_{l \in L_{ijk}} \begin{bmatrix} W_{ijkl} \\ h_{ijkl}(x) \le 0 \end{bmatrix} \bigvee \begin{bmatrix} W_{ijk0} \\ x^{lo} \le x \le x^{up} \end{bmatrix} \quad \forall i \in I, j \in J_i, k \in K_{ij}$$

Logical Propositions

$$\veebar_{j \in J_i} Y_{ij} \quad \forall i \in I$$

New XOR proposition

$$\veebar_{l \in L_{ijk}} W_{ijkl} \veebar W_{ijk0} \quad \forall i \in I, j \in J_i, k \in K_{ij}$$

$$\veebar_{l \in L_{ijk}} W_{ijkl} \Leftrightarrow Y_{ij} \quad \forall i \in I, j \in J_i$$

$$\Omega(Y, W) = True$$

Variable Domains

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \quad \forall i \in I, j \in J_i$$

$$W_{ijkl} \in \{True, False\} \quad \forall i \in I, j \in J_i, \forall k \in K_{ij}, l \in L_{ijk} \cup \{0\}$$

The proposed approach is to explicitly model GDPs with the nested disjunctions (*Approach 2*):

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

Nested Disjunctions

$$\bigvee_{j \in J_i} \begin{bmatrix} Y_{ij} \\ g_{ij}(x) \le 0 \\ \bigvee_{l \in L_{ijk}} \begin{bmatrix} W_{ijkl} \\ h_{ijkl}(x) \le 0 \end{bmatrix} \quad \forall k \in K_{ij} \end{bmatrix} \quad \forall i \in I$$

Logical Propositions

$$\veebar_{j \in J_i} Y_{ij} \quad \forall i \in I$$

$$\veebar_{l \in L_{ijk}} W_{ijkl} \Leftrightarrow Y_{ij} \quad \forall i \in I, j \in J_i$$

$$\Omega(Y, W) = True$$

Variable Domains

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \quad \forall i \in I, j \in J_i$$

$$W_{ijkl} \in \{True, False\} \quad \forall i \in I, j \in J_i, \forall k \in K_{ij}, l \in L_{ijk}$$

NGDP Reformulation Approach 1 (Equivalent Single-level GDP Reformulation)

$$\bigvee_{j\in J}\left[\begin{array}{c} Y_j \\ g_j(x)\le 0 \\ \bigvee_{k\in K_j}\left[\begin{array}{c} W_{jk} \\ h_{jk}(x)\le 0\end{array}\right]\end{array}\right]$$

Extract Inner Disjunction

$$\bigvee_{j\in J}\left[\begin{array}{c} Y_j \\ g_j(x)\le 0\end{array}\right]$$

additional disjunct

$$\bigvee_{k\in K_j}\left[\begin{array}{c} W_{jk} \\ h_{jk}(x)\le 0\end{array}\right]\bigvee\left[\begin{array}{c} W_{j0} \\ x^{lo}\le x\le x^{up}\end{array}\right]\quad \forall j\in J$$

The additional disjunct will incur the creation of a <u>new binary variable</u> (ω_{j0}). A <u>new disaggregated variable</u> (μ_{j0}) is also created in the hull reformulation

Big-M Reformulation (BM1)

$$g_j(x)\le M_j\left(1-\lambda_j\right)\quad \forall j\in J$$

$$h_{jk}(x)\le m_{jk}\left(1-\omega_{jk}\right)\quad \forall j\in J, k\in K_j$$

$$x^{lo}-\mathcal{M}_{jk}\left(1-\omega_{j0}\right)\le x\le x^{up}+\mathcal{M}_{jk}\left(1-\omega_{j0}\right)$$

Hull Reformulation (HR1)

$$x=\sum_{j\in J}\nu_j$$

$$\lambda_j g_j\left(\frac{\nu_j}{\lambda_j}\right)\le 0\quad \forall j\in J$$

additional slack

$$x=\sum_{k\in K_j}\mu_{jk}+\mu_{j0}$$

$$\omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0\quad \forall j\in J, k\in K_j$$

NGDP Reformulation Approach 2 (Direct NGDP Reformulation)

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ \bigvee_{k\in K_j}\begin{bmatrix} W_{jk} \\ h_{jk}(x)\le 0\end{bmatrix}\end{bmatrix}$$

Hull of Inner Disjunction →

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ x=\sum_{k\in K_j}\mu_{jk} \\ \omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0 \quad \forall j\in J, k\in K_j\end{bmatrix}$$

Hull Reformulation (HR2) ↓

$$x=\sum_{j\in J}\nu_j$$

$$\lambda_j g_j\left(\frac{\nu_j}{\lambda_j}\right)\le 0 \quad \forall j\in J$$

$$\nu_j=\sum_{k\in K_j}\mu_{jk}$$

$$\omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0 \quad \forall j\in J, k\in K_j$$

Inner disjunction variables are disaggregated only once

Big-M of Inner Disjunction ↓

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ h_{jk}(x)\le m_{jk}\cdot(1-\omega_{jk}) \quad \forall k\in K_j\end{bmatrix}$$

Big-M Reformulation (BM2) ↓

$$g_j(x)\le M_j(1-\lambda_j) \quad \forall j\in J$$

$$h_{jk}(x)\le m_{jk}\cdot(1-\omega_{jk}) + M_j(1-\lambda_j) \quad \forall j\in J, k\in K_j$$

2 big-M parameters:

- m: inner disjunction

- M:outer disjunction

Big-M Reformulation Tightness

Theorem 1. The continuous relaxation of the **Direct NGDP Big-M Reformulation** (Approach 2 Big-M Reformulation; *r-BM2*) is as tight as the continuous relaxation of the **Equivalent Single-level GDP Reformulation** (Approach 1 Big-M Reformulation; *r-BM1*) when the tightest big-M values are used:

r-BM2* ⊆ *r-BM1

Proof: The two approaches differ in the Big-M reformulation of the constraints in the inner disjunct:

Approach 1: $h_{jk}(x) \leq m_{jk}(1 - \omega_{jk}) \quad \forall j \in J, k \in K_j$

Approach 2: $h_{jk}(x) \leq m'_{jk} \cdot (1 - \omega_{jk}) + M'_j(1 - \lambda_j) \quad \forall j \in J, k \in K_j$

Let $h_{jk}^{max} = \max\{h_{jk}(x) | x \in R^n, x^{lo} \leq x \leq x^{up}\}$ be the maximum value of the constraint $h_{jk}(x)$ in the feasible space of x. The tightest Big-M values satisfy:

Approach 1: $m_{jk} = h_{jk}^{max}$

Approach 2: $m'_{jk} + M'_j = h_{jk}^{max}$

Since $\lambda_j \geq \omega_{jk}$, the right-hand side of the Big-M constraint in Approach 2 can be shown to be as tight as the right-hand side of the Big-M constraint in Approach 1:

$$m'_{jk} \cdot (1 - \omega_{jk}) + M'_j(1 - \lambda_j) \leq m_{jk}(1 - \omega_{jk})$$

Hull Reformulation Tightness

Theorem 2. The continuous relaxation of the **Direct NGDP Hull Reformulation** (Approach 2 Hull Reformulation; *r-HR2*) is as tight as the continuous relaxation of the **Equivalent Single-level GDP Reformulation** (Approach 1 Hull Reformulation; *r-HR1*):

r-HR2* ⊆ *r-HR1

Proof: Use Fourier-Motzkin to eliminate the additional disaggregated variable (μ_{j0}) and its associated binary (ω_{j0}) resulting from the additional disjunct (W_{j0}) created when extracting the inner disjunct in Approach 1. Note the following,

- W_{j0} is selected iff the main disjunct is not (Y_j): $\omega_{j0} = 1 - y_j$.
- μ_{j0} is bounded by $0 \leq \mu_{j0} \leq x^{up}\omega_{j0}$.

The elimination yields the constraint,

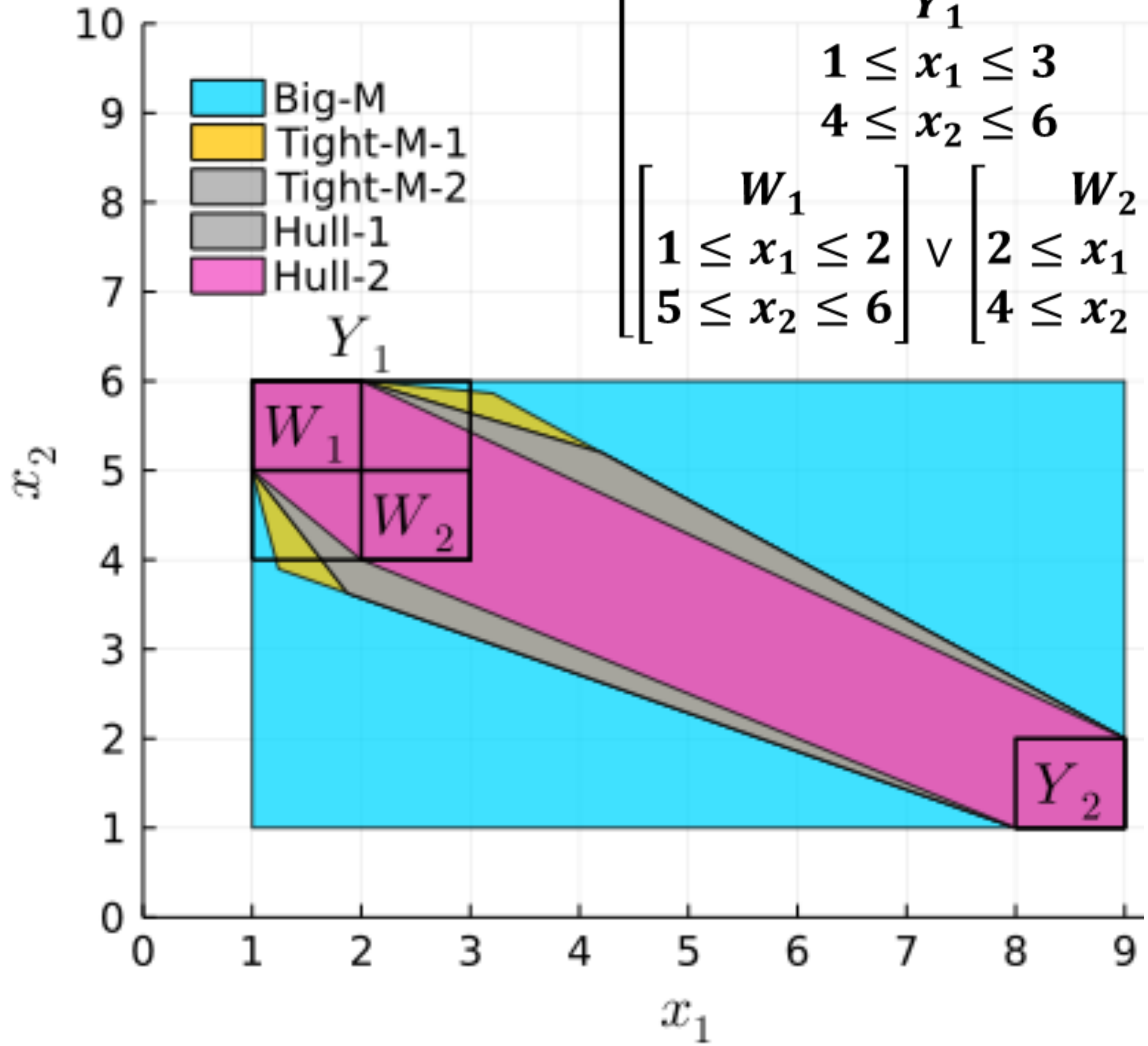
$$\nu_j + \sum_{j' \in J : j' \neq j} \nu_{j'} - x^{up}(1 - y_j) \leq \sum_{k \in K_j} \mu_{jk} \leq \nu_j + \sum_{j' \in J : j' \neq j} \nu_{j'}$$

which can be shown to be a relaxation of the analogous constraint in Approach 2,

$$\nu_j \leq \sum_{k \in K_j} \mu_{jk} \leq \nu_j$$

Note: $0 \leq \sum_{j' \in J : j' \neq j} \nu_{j'} \leq x^{up}(1 - y_j)$

Reformulation Tightness Example



$$\begin{bmatrix} Y_1 \\ 1 \le x_1 \le 3 \\ 4 \le x_2 \le 6 \\ \begin{bmatrix} W_1 \\ 1 \le x_1 \le 2 \\ 5 \le x_2 \le 6 \end{bmatrix} \vee \begin{bmatrix} W_2 \\ 2 \le x_1 \le 3 \\ 4 \le x_2 \le 5 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} Y_2 \\ 8 \le x_1 \le 9 \\ 1 \le x_2 \le 2 \end{bmatrix}$$

- **Big-M** = Big-M reformulation for Approach 1 or Approach 2
- **Tight-M-1** = Big-M reformulation for Approach 1 with tightest M values
- **Tight-M-2** = Big-M reformulation for Approach 2 with tightest M values
- **Hull-1** = Hull reformulation for Approach 1
- **Hull-2** = Hull reformulation for Approach 2

Nested GDP Example: Design & Process Scheduling

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Sales ***Purchases*** ***Installation & Operating Costs***

$$\max \quad Z = \sum_{t\in T}[p_C F_{8,t} - (p_B F_{3,t} + p_A F_{0,t})] - \sum_{r\in\mathcal{R}} IC_r - \sum_{i\in I} IC_i - \sum_{t\in T}\sum_{i\in I} OC_{i,t}$$

Tank levels

$s.t.$

$$R_{A,t} = R_{A,t-1} + F_{0,t} - F_{1,t} \quad \forall t\in T$$

$$R_{B,t} = R_{B,t-1} + F_{2,t} + F_{3,t} - F_{4,t} + F_{5,t} \quad \forall t\in T$$

$$R_{C,t} = R_{C,t-1} + F_{6,t} + F_{7,t} - F_{8,t} \quad \forall t\in T$$

Reactor availability

$$R_{U,t} = R_{U,t-1} - \sum_{i\in I}\Delta R_{i,t} + \sum_{i\in I}\Delta R_{i,t-\tau_i} \quad \forall t\in T$$

Tank installation (X)

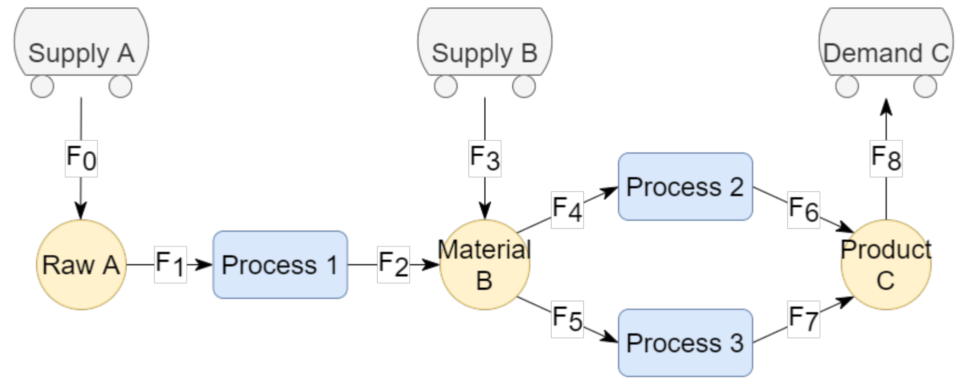
$$\begin{bmatrix} X_r \\ IC_r = \alpha_r + \beta_r Q_r \end{bmatrix} \vee \begin{bmatrix} \neg X_r \\ IC_r = 0 \\ Q_r = 0 \end{bmatrix} \quad \forall r\in\mathcal{R}$$

Process selection (Y), technology selection (W)

$$\begin{bmatrix} Y_i \\ \begin{bmatrix} W_{i,1} \\ IC_i = \alpha_{i,1} + \beta_{i,1} Q_i \end{bmatrix} \vee \begin{bmatrix} W_{i,2} \\ IC_i = \alpha_{i,2} + \beta_{i,2} Q_i \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \neg Y_i \\ Q_i = 0 \\ IC_i = 0 \end{bmatrix} \quad \forall i\in I$$

Schedule production (N)

$$\begin{bmatrix} N_{i,t} \\ B_{i,t} \le Q_i \\ \Delta R_{i,t} = 1 \\ F_{in,t} = B_{i,t} \\ \begin{bmatrix} W_{i,t,1} \\ F_{out,t+\tau_i} = \nu_{i,1} B_{i,t} \\ OC_{i,t} = \gamma_{i,1} B_{i,t} \end{bmatrix} \vee \begin{bmatrix} W_{i,t,2} \\ F_{out,t+\tau_i} = \nu_{i,2} B_{i,t} \\ OC_{i,t} = \gamma_{i,2} B_{i,t} \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \neg N_{i,t} \\ B_{i,t} = 0 \\ \Delta R_{i,t} = 0 \\ F_{in,t} = 0 \\ F_{out,t+\tau_i} = 0 \\ OC_{i,t} = 0 \end{bmatrix} \quad \forall i\in I, t\in T$$



Logic Propositions

Select technology for installation

$$Y_i \Leftrightarrow W_{i,1} \underline{\vee} W_{i,2} \quad \forall i\in I$$

Select operating technology

$$N_{i,t} \Leftrightarrow W_{i,t,1} \underline{\vee} W_{i,t,2} \quad \forall i\in I, t\in T$$

Match operating & installed technology

$$W_{i,t,m} \Rightarrow W_{i,m} \quad \forall i\in I, m\in M_i$$

Iyer and Grossmann (1998)

Nested GDP Example Results: Design & Process Scheduling

	Big-M Reformulation (Approach 1)	Big-M Reformulation (Approach 2)	Hull Reformulation (Approach 1)	Hull Reformulation (Approach 2)
Model Size	**171 binaries** **236 continuous** **1,338 constraints**	**138 binaries** **232 continuous** **1,235 constraints**	**171 binaries** **770 continuous** **3,161 constraints**	**138 binaries** **796 continuous** **3,199 constraints**
Relaxation Gap	**94.2%**	**89.5%**	**5%**	**4%**
Relaxation Simplex Iterations	179	211	195	154
Nodes Explored	**562**	**2,326**	**0**	**0**
Solution Time	0.68 s	1.14 s	0.22 s	0.49 s

Implementation: JuMP 1.0
Disjunctive Programming: 0.3
Solver: CPLEX 20.1 (default options)

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Optimization model for power generation and transmission expansion planning

Given: Generation sources, load demand for each region, CO_2 emission limits and economic data

Coal power plants

Natural gas power plants

Nuclear power plants

Wind turbines

Solar panels

***Determine*:** Number and type of generators / transmission lines, Unit commitments, Total cost

***Goal*: Long term Planning to Minimize Total Cost**

Method: Large-scale multiperiod MILP model (Benders decomposition)

Limitation:

Reliability (i.e., generators or transmission lines failures) are not explicitly considered

***Generation**:* Lara, C. L., Mallapragada, D. S., Papageorgiou, D. J., Venkatesh, A., & Grossmann, I. E. (2018). Deterministic electric power infrastructure planning: Mixed-integer programming model and nested decomposition algorithm. *European Journal of Operational Research*, 271, 1037-1054.

Generation and Transmission: Li, C., A.J. Conejo, P. Liu, B.P. Omell, J.D. Siirola, I.E. Grossmann. (2021) Mixed-integer Linear Programming Models and Algorithms for Generation and Transmission Expansion Planning of Power Systems. *European Journal of Operational Research,* 297, 1071-1082.

Problem statement

Seolhee Cho

Goal

Develop a Generalized Disjunctive Programming model for multi-period and multi-site capacity planning of reliable power generation systems

Characteristics of the model

- Dual role of backup generators depending on power demand
- Advanced probabilistic model for reliability evaluation considering redundancy

Given

- Regions, power stations, backup generators
- Economic data (investment & operating cost)
- Unit reliability and electricity demand
- CO_2 emission and carbon tax

Determine

- When and where to install and retire the generators?
- Number and size of backup generators
- System reliability & expected power production

Overview of GDP optimization model

Summary of constraints

- **Balance of power plant *k*** that are available, decommissioned, and extend lifetime
- Balance and capacity of newly installed and available backup generator *j* in time *t*
- Available backup generators *j* in year *t* can either operate or remain as backup during sub-period *n* of year *t*
- Total feedstock, total expected power output, CO_2 emission, and symmetry breaking constraints
- LOLE and LOEE penalties calculation
- **Failure state probability, and corresponding expected power output** under specific design *h* and operation mode *m*

  → These are calculated based on **embedded disjunctions**

  → Two Boolean variables: $Z_{k,r,h,t}$ – Planning decision, $W_{k,r,m,h,t,n}$ – Operation decision

Nested Disjunction in GDP formulation

$$\left[\bigvee_{c\in C_k}\left[\begin{array}{c} X^{OP}_{j,k,r,t,n} \\ X^{BOP}_{j,c,k,r,t,n} \\ OC^{BAK}_{j,c,k,r,t,n} = \psi_{j,c,k} C^{BOP}_{j,c,k,r,t,n} \\ \gamma^{min}_{j,k}\varphi_{j,c,k} \le C^{BOP}_{j,c,k,r,t,n} \le \gamma^{max}_{j,k}\varphi_{j,c,k} \end{array}\right]\right] \vee \left[\begin{array}{c} \neg X^{OP}_{j,k,r,t,n} \\ OC^{BAK}_{j,c,k,r,t,n} = 0 \\ C^{BOP}_{j,c,k,r,t,n} = 0 \end{array}\right]$$

$$\forall\, j \in J_k, k \in K_r, r \in R, t \in T, n \in N$$

$$\left[\begin{array}{c} X^{PO}_{k,r,t,n} \\ OC^{P}_{k,r,t,n} = \delta_k C^{PO}_{k,r,t,n} \\ \gamma^{min}_k \omega_k \le C^{PO}_{k,r,t,n} \le \gamma^{max}_k \omega_k \\ 0 \le P^F_{k,r,t,n} \le 1 \\ 0 \le EP_{s,k,r,t,n} \le \overline{M_k} \quad \forall s \in S \end{array}\right] \vee \left[\begin{array}{c} \neg X^{PO}_{k,r,t,n} \\ OC^{P}_{k,r,t,n} = 0 \\ C^{PO}_{k,r,t,n} = 0 \\ P^F_{k,r,t,n} = 1 \\ EP_{s,k,r,t,n} = 0 \end{array}\right] \quad \forall k \in K_r, r \in R, t \in T, n \in N$$

$$X^{OP}_{j,k,r,t,n} \Leftrightarrow \bigvee_{c\in C_k} X^{BOP}_{j,c,k,r,t,n} \quad \forall\, j \in J_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{BOP}_{j,c,k,r,t,n} \Rightarrow Y^{BAV}_{j,c,k,r,t} \quad \forall\, j \in J_k, c \in C_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{BOP}_{j,c,k,r,t,n} \Rightarrow X^{PO}_{k,r,t,n} \quad \forall\, j \in J_k, c \in C_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{PO}_{k,r,t,n} \Rightarrow Y^{PA}_{k,r,t} \quad \forall\, k \in K_r, r \in R, t \in T, n \in N$$

$$\bigvee_{h\in H_k}\left[\begin{array}{c} Z_{k,r,h,t} \\ \bigvee_{m\in M_h}\left[\begin{array}{c} W_{k,r,m,h,t,n} \\ P^F_{k,r,t,n} = \sum_{s\in S^F_{m,h}} P_{s,k,r} \\ EP_{s,k,r,t,n} = P_{s,k,r}\left(C^{PO}_{k,r,n,t} + \sum_{j\in J^O_{s,m,h}} C^{OP}_{j,k,r,t,n}\right) \quad \forall s \in S^F_{m,h} \\ EP_{s,k,r,t,n} = P_{s,k,r}\left(C^{PA}_{k,r,t} + \sum_{j\in J^O_{s,m,h}} C^{AV}_{j,k,r,t}\right) \quad \forall s \in S^P_{m,h} \end{array}\right] \forall n \in N \end{array}\right] \quad \forall k \in K_r, r \in R, t \in T$$

$Z_{k,r,h,t}$ → Design decisions in time *t*

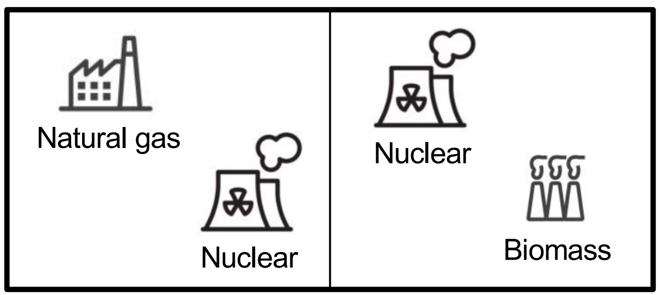
$W_{k,r,m,h,t,n}$ → Operation decisions during subperiod *n* in time *t*

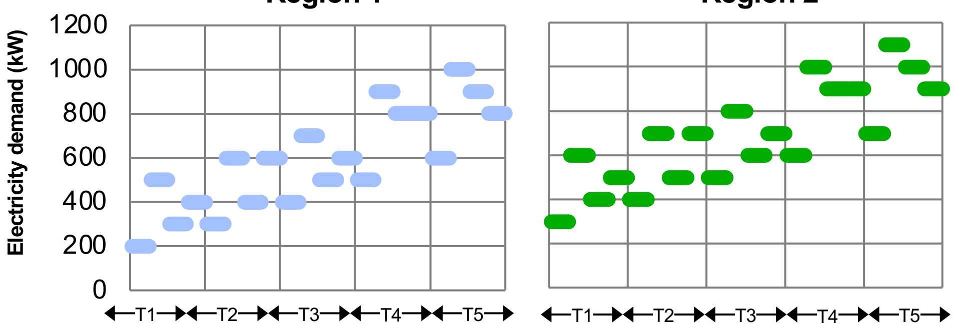
Successful reliability $\quad P_{s,k,r} = \prod_{j\in J^A_{s,h}} \lambda_{j,k} \prod_{j\in J^F_{s,h}} (1-\lambda_{j,k}) \quad \forall s \in S_h, |S| = 2^{|h|}, k \in K_r, r \in R$

Expected production level depending on design and operation

Illustrative example

- Two regions (R1, R2) and three different power technologies are installed (natural gas, nuclear, and biomass gasification)
- By adding backup generators, each power plant can both expand its capacity and improve reliability





	NG/Nuclear/Biomass	Three capacities for backup generators	small/medium/large
Unit reliability (%)	90 / 95 / 95	Capacity size of backup generator (kW)	100 / 200 / 300
Installed capacity of power plant (kW)	200 / 300 / 200	Installation cost of NG by size (k$/kW)	5 / 8 / 10
Operating cost ($/kW)	20 / 40 / 15	Installation cost of nuclear by size (k$)	15 / 20 / 25
Conversion efficiency (%)	40 / 45 / 40	Installation cost of biomass by size (k$)	2.5 / 4.5 / 6
Feedstock cost ($/MMBtu)	0.3 / 0.01 / 0.2	Average CO_2 emission cost ($/kg)	3
Operating cost of power plant ($/kW)	20 / 40 / 15		

LOEE (unmet demand) penalty rate ($/kWh)	10	LOEE (downtime) penalty rate ($/hour)	5,000

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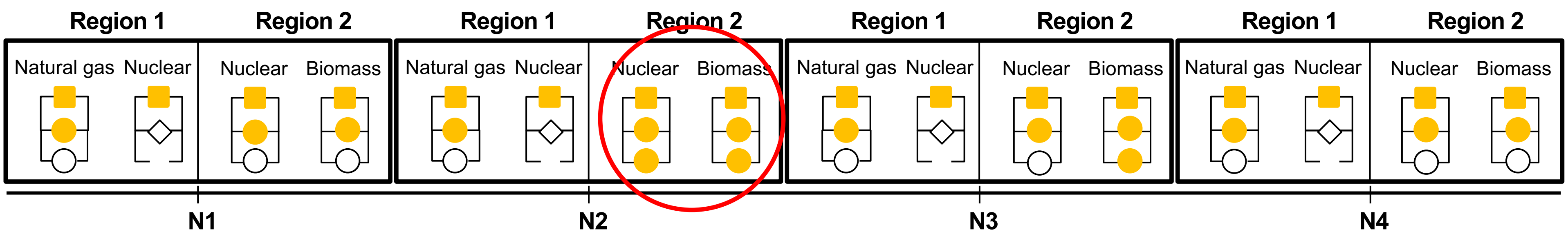
Illustrative example:

2 regions: Natural gas/Nuclear, Nuclear/Biomass over 5 year

Operation result: 4 demand periods, year 5

Discrete capacity (kW) ◇ 100 △ 200 ○ 300

○ Backup generator
● Operating generator



- **Backup generator are used to produce more power (Region 2/2nd demand period)**
- Reliable design and operation strategies are highly dependent by LOLE and LOEE penalties

Computational results

Approach 2	Constraints	Cont. Variables	Binary variables	CPU (sec)	Rel gap%	LP relaxation (k$)	Cost (k$)
Big-M	12,419	4,631	1,644	1,668.31	84.0	115.7	722.21
Hull reformulation	23,503	12,435	1,644	10.84	11.8	636.9	722.21

- Solver: Gurobi 32.1.0
- Tight bounds of convex hull reformation → Short computational time

Conclusions

- Extending GDP to allow ***nested disjunctions*** allows explicitly modeling problems with multiple hierarchies
- Two approaches to reformulate NGDPs into MI(N)LPs have been formalized:
  - Approach 1: transforming into an equivalent single-level GDP and then reformulating
  - Approach 2: reformulating NGDP from the inside-out
- Relaxations of reformulated models via Approach 2 are shown to be as tight or tighter than their counterparts obtained via Approach 1
- Examples show superiority Approach 2

Remarks

NGDPs and their reformulations to MI(N)LPs can be extended to **multi-level hierarchies**:

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

Level 1 ***Level 2*** ***Level N***

$$\bigvee_{j_1 \in J^{(1)}_{i_1}} \left[\begin{array}{c} Y^{(1)}_{i_1 j_1} \\ g^{(1)}_{i_1 j_1}(x) \le 0 \\ \bigvee_{j_2 \in J^{(2)}_{i_1 j_1, i_2}} \left[\begin{array}{c} Y^{(2)}_{i_1 j_1, i_2 j_2} \\ g^{(2)}_{i_1 j_1, i_2 j_2}(x) \le 0 \\ \vdots \\ \left[\bigvee_{j_n \in J^{(N)}_{i_1 j_1, \dots, i_{N-1} j_{N-1}, i_N}} \begin{bmatrix} Y^{(N)}_{i_1 j_1, \dots, i_N j_N} \\ g^{(N)}_{i_1 j_1, \dots, i_N j_N}(x) \le 0 \end{bmatrix} \quad \forall i_N \in I^{(N)}_{i_1 j_1, \dots, i_{N-1} j_{N-1}} \right] \end{array} \right] \quad \forall i_2 \in I^{(2)}_{i_1 j_1} \end{array} \right] \quad \forall i_1 \in I^{(1)}$$

$$\Omega(Y) = True$$

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y^{(n)}_{i_1 j_j, \dots, i_k j_k} \in \{True, False\} \quad \forall n \in \{1, \dots, N\}$$