Sequential penalty methods for mixed integer programs

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MIP 2022

Mixed-Binary Linear Complementarity Problems

The mixed-binary linear complementarity problem (MILCP) is the task to **find a vector $z \in \mathbb{R}^n$ that satisfies**

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^\top (q + Mz) &= 0 \\ z_i &\in \{0, 1\} \qquad \text{for } i \in I \subseteq \{1, \ldots, n\} \end{aligned}$$

or to show that no such vector exists, for given

- $M \in \mathbb{R}^{n \times n}$, $M \succeq 0$
- $q \in \mathbb{R}^n$

Application context

Linear Complementarity Problems (LCPs) are an important tool for the modeling and analysis of equilibrium problems in economics, mechanics, ... [Cottle, Pang, Stone; "The Linear Complementarity Problem"; 2009]
[Gabriel, Conejo, Fuller, Hobbs; "Complementarity modeling in energy markets"; 2012]

When a subset of variables is restricted to take integer values, i.e., $z_i \in \mathbb{Z}$ for a given index set $I \subseteq \{1, \ldots, n\}$ we fall in the context of MILCPs

A common tool in the analysis and resolution of a Linear Complementarity Problem (LCP) is its reformulation as Quadratic Problem (QP) [Cottle et al.;2009]:

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^\top (q + Mz) &= 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & z^\top (q + Mz) \\ \text{s.t.} \quad & q + Mz \geq 0 \\ & z \geq 0 \end{aligned}$$

Linear Complementarity Problems

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LCP has a solution if and only if the QP has an optimal solution with objective function value zero

MIQP reformulation of a MILCP

Equivalently we can reformulate a MILCP into a MIQP:

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^\top (q + Mz) &= 0 \\ z_i \in \{0, 1\},\ i &\in I \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min \quad & z^\top (q + Mz) \\ \text{s.t.} \quad & q + Mz \geq 0 \\ & z \geq 0 \\ & z_i \in \{0, 1\},\ i \in I \end{aligned}$$

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$$

MILCP has a solution if and only if the MIQP
has an optimal solution with objective function value zero

However, the existence of a solution of the MILCP cannot be expected in general...

...look for "approximate feasible solutions"

For practically relevant instances where non-existence occurs, one is interested in **"approximate feasible solutions"**:

points that minimize a certain infeasibility measure that combines both the violation of integrality conditions as well as of complementarity constraints

Penalizing the violation of complementarity and integrality

$$
\begin{aligned}
\min \quad & \alpha\, P_C(z) + (1-\alpha)\, P_I(z) \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \geq 0 \\
& z_i \leq 1, \quad i \in I
\end{aligned}
$$

where

- $\alpha \in [0,1]$
- $P_C(z)$ is a function penalizing the violation of the complementarity constraints
- $P_I(z)$ is a function penalizing the violation of the integrality constraints

  [Raghavachari;1969], [Giannessi, Tardella; 1998], [Zhu; 2003], [Lucidi, Rinaldi; 2010], [De Santis, Lucidi, Rinaldi; 2013]

A nonconvex, nonsmooth reformulation of MILCP

$$\begin{aligned} \min \quad & \alpha\, z^\top(q + Mz) + (1-\alpha) \textstyle\sum_{i \in I} \min\{z_i, 1 - z_i\} \\ \text{s.t.} \quad & q + Mz \geq 0 \\ & z \geq 0 \\ & z_i \leq 1, \quad i \in I \end{aligned} \qquad (\mathrm{NC}_{ref})$$

where

- $\alpha \in [0, 1]$
- $P_C(z) = z^\top(q + Mz)$
- $P_I(z) = \sum_{i \in I} \min\{z_i, 1 - z_i\}$

  $P_I(z)$ **is concave and piecewise linear**

Features of the *penalty* branch-and-bound method

In order to globally solve problem NC_{ref}, we address a **sequence of convex quadratic smooth problems** that

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$$\Downarrow$$

the objective function slightly changes along the iterations!

Problem at the root node

At the **root node** of the branch-and-bound tree, we solve the convex smooth problem

$$\begin{aligned} \min \quad & \alpha\, z^\top (q + Mz) \\ \text{s.t.} \quad & q + Mz \geq 0 \\ & z \geq 0 \\ & z_i \leq 1, \quad i \in I \end{aligned}$$

obtained from Problem (NC_{ref}) by **neglecting the second term in the objective function**

Branching

Let z^* be the solution of the root node relaxation

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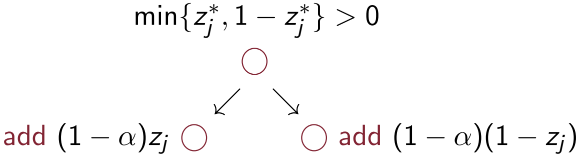

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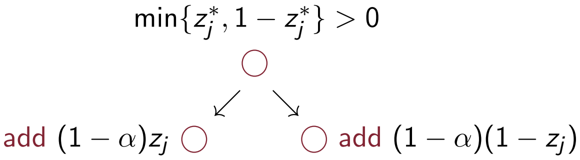

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\begin{aligned}
\min \quad & \alpha\, z^\top (q + Mz) + (1-\alpha) z_j \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \geq 0, \quad z_i \leq 1, \quad i \in I
\end{aligned}
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⇢ *aims to drive z_j to 0*
in the respective subtree

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Choose an index $j \in I$ such that $\min\{z_j^*, 1 - z_j^*\} > 0$ and **build two children nodes**:

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add $(1-\alpha)z_j$ ⟵ ○ ⟶ add $(1-\alpha)(1-z_j)$

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$\dashrightarrow$ *aims to drive z_j to 0 in the respective subtree*

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$\dashrightarrow$ *aims to drive z_j to 1 in the respective subtree*

Problem at the node $N = (I_0, I_1)$

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The subproblem at node $N = (I_0, I_1)$ is

$$
\begin{aligned}
\min \quad & f_N(z) \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \geq 0 \\
& z_i \leq 1, \quad i \in I
\end{aligned}
$$

with

$$f_N(z) = \alpha z^\top (q + Mz) + (1-\alpha)\left(\sum_{j \in I_0} z_j + \sum_{j \in I_1} (1 - z_j)\right)$$

Enumerating the partitions (I_0, I_1) of $I \subseteq \{1, \ldots, n\}$

The minimum among the optimal solutions of the problems of **all leaf nodes** of the fully enumerated branch-and-bound tree is the optimal solution of Problem (NC_{ref}):

Lemma

Let z^* be an optimal solution of Problem (NC_{ref}) and z_N^* the solution at the node $N = (I_0, I_1)$. Then,

$$f(z^*) = \min\{f_N(z_N^*) \colon N = (I_0, I_1) \text{ with } I_0 \cup I_1 = I \text{ and } I_0 \cap I_1 = \emptyset\}$$

Bounding and Pruning

The optimal value $f_N(z_N^*)$ of the problem defined at a node N is a local lower bound for the subtree rooted in N:

Lemma

Let $N' = (I_0', I_1')$ be a successor of $N = (I_0, I_1)$, i.e., $I_0 \subseteq I_0'$ and $I_1 \subseteq I_1'$. Then,

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and **we can prune the subtree rooted in N**

MILCP-PBB Scheme

Input: $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$
Output: A global optimum z^* of Problem (NC_{ref})

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        Choose $j \in I \setminus (I_0 \cup I_1)$
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return z^*_{inc}

Finite termination

Theorem

Algorithm `MILCP-PBB` terminates after finitely many steps with a global optimal solution of Problem (NC_{ref}).

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Remark

Note that in our branch-and-bound method, there is no direct analogy to pruning due to infeasibility.

In case at a node we find a feasible solution for the MILCP we stop the algorithm

Adding simple cuts

Within the node subproblem we include simple bound constraints:

$$
\begin{aligned}
\min \quad & f_N(z) \\
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Lemma

Let z_N^* be an optimal solution at node N when simple cuts are included. Then,

$$f(z^*) = \min \{ f_N(z_N^*) : N = (I_0, I_1) \textit{ with } I_0 \cup I_1 = I \}$$

Adding simple cuts

finite termination

Lemma

Let $N' = (I'_0, I'_1)$ be a successor of some node $N = (I_0, I_1)$ in the branching tree, i.e., $I_0 \subseteq I'_0$ and $I_1 \subseteq I'_1$ holds. Further, let $z^*_N, z^*_{N'}$ be optimal solutions of nodes N and N', respectively, when simple cuts are used. Then,

$$f_N(z^*_N) \leq f_{N'}(z^*_{N'})$$

holds.

Theorem

Algorithm `MILCP-PBB` remains correct when simple cuts

$$z_j \leq 0.5 \text{ for all } j \in I_0, \quad z_j \geq 0.5 \text{ for all } j \in I_1$$

are added at any node $N = (I_0, I_1)$.

Numerical results

Randomly generated instances

We built matrices $M \in \mathbb{R}^{n \times n}$ with $n \in \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500\}$.

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(a) only $q + Mz \geq 0, \; z \geq 0$ are guaranteed to be satisfied,

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(c) only $q + Mz \geq 0, \ z \geq 0$ and complementarity $({z^*}^\top (q + Mz^*) = 0)$ are guaranteed to be satisfied,

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(c) only $q + Mz \geq 0, \; z \geq 0$ and complementarity $({z^*}^\top(q + Mz^*) = 0)$ are guaranteed to be satisfied,

We created 10 instances for every size n and the types (a)–(c), yielding 300 different instances in total.

Numerical results on the use of simple cuts

Performance Profiles

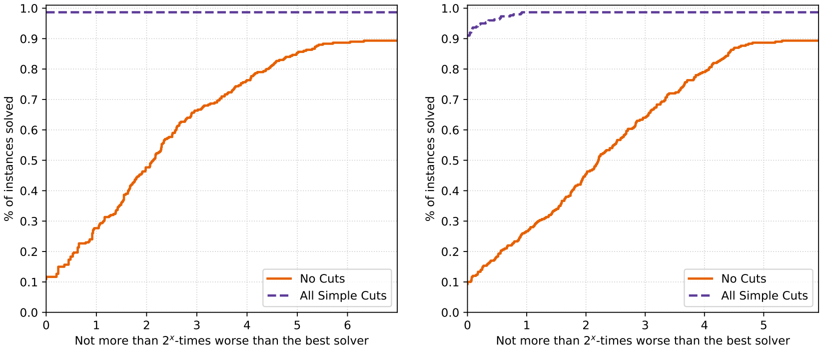

Figure: Performance profiles: number of nodes (left), running time (right)

Numerical comparison on branching rules

Performance Profiles

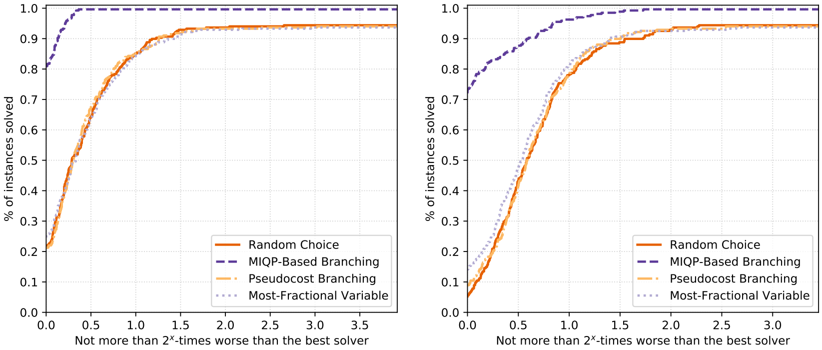

Figure: Performance profiles: number of nodes (left), running time (right)

MIQP-based branching rule

We presolve single-binary-variable MIQPs, one for each z_j, $j \in I$:

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} \quad & z^\top (q + Mz) && \text{(1a)} \\
\text{s.t.} \quad & q + Mz \geq 0, \; z \geq 0, && \text{(1b)} \\
& z_j \in \{0, 1\}. && \text{(1c)}
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measuring the impact of the j-th variable on the infeasibility of the problem

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We sort the indices $j \in I$ in decreasing order with respect to the optimal objective function values

Comparison with other approaches

An MILP reformulation, with additional binary variables and big-M constraints

[Gabriel, Conejo, Ruiz, Siddiqui; 2013]

$$
\begin{aligned}
\min_{z,z',z'',\rho,\sigma} \quad & \alpha \sum_{i=1}^{n} \rho_i + (1-\alpha) \sum_{i \in I} \sigma_i && \text{(2a)}\\
\text{s.t.} \quad & z \geq 0, \quad q + Mz \geq 0, && \text{(2b)}\\
& z \leq Bz' + \rho, && \text{(2c)}\\
& q + Mz \leq B(1 - z') + \rho, && \text{(2d)}\\
& 0 \leq z_I \leq z'' + \sigma, && \text{(2e)}\\
& z'' - \sigma \leq z_I \leq 1, && \text{(2f)}\\
& z \in \mathbb{R}^n, \quad z' \in \{0,1\}^n, \quad z'' \in \{0,1\}^I, && \text{(2g)}\\
& \sigma \in \mathbb{R}^I_{\geq 0}, \quad \rho \in \mathbb{R}^n_{\geq 0}. && \text{(2h)}
\end{aligned}
$$

variables: $3n + 2|I|$, $(n + |I|$ constrained to be binary)

Comparison with other approaches

An MIQP reformulation, no big-M constraints

$$\min_{z,z',\sigma} \quad \alpha z^\top (q + Mz) + (1-\alpha) \sum_{i \in I} \sigma_i \tag{3a}$$

$$\text{s.t.} \quad z \geq 0, \quad q + Mz \geq 0, \tag{3b}$$

$$0 \leq z_I \leq z' + \sigma, \tag{3c}$$

$$z' - \sigma \leq z_I \leq 1, \tag{3d}$$

$$z \in \mathbb{R}^n, \quad z' \in \{0,1\}^I, \tag{3e}$$

$$\sigma \in \mathbb{R}^I_{\geq 0}. \tag{3f}$$

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Comparison with GUROBI addressing the MILP and the MIQP reformulations

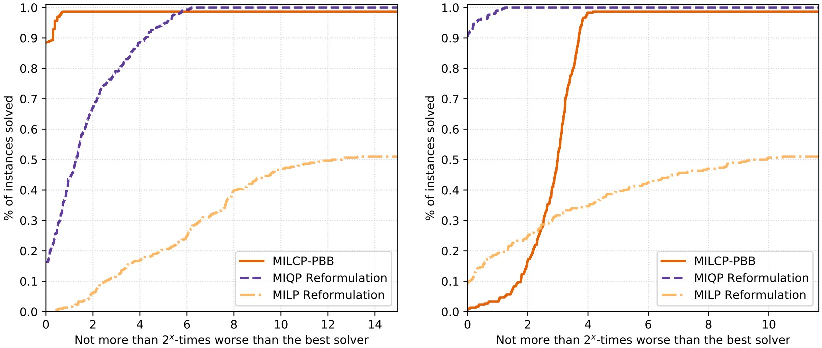

Figure: Performance profiles: number of nodes (left), running time (right)

Comparison with GUROBI addressing the MIQP

Harder test set (300 instances with $n = 100, \ldots, 600$)

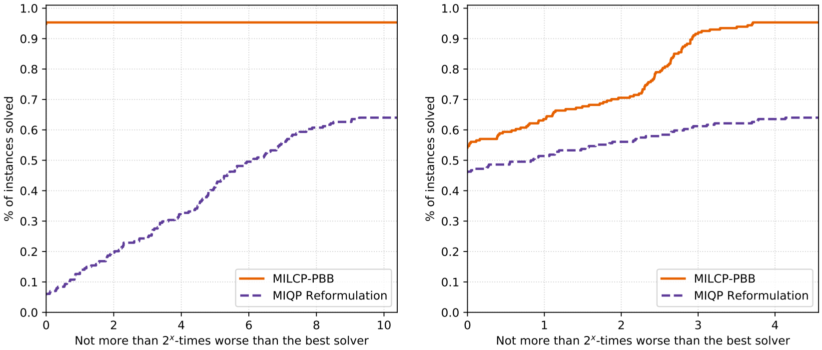

Figure: Performance profiles: number of nodes (left), running time (right)

Conclusions

We presented a **penalty branch-and-bound method** for MILCPs

- the method is able to compute a solution if one exists or it computes an approximate solution that minimizes an infeasibility measure based on the violation of the integrality and complementarity conditions of the problem
- the objective function slightly changes along the nodes so that the penalization of the integrality constraint violation is progressively increased

Future work

...useful for MILPs?

Under specific assumption on $P = \{x \in \mathbb{R}^n : Ax \le b\}$ we can prove that $\epsilon > 0$ exists such that

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & x_i \in \{0,1\}, \quad i \in I \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min \quad & c^\top x + \tfrac{1}{\epsilon} \textstyle\sum_{i \in I} \min\{x_i, 1 - x_i\} \\ \text{s.t.} \quad & Ax \le b \\ & x \in [0,1]^n \end{aligned}$$

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we can use our branch-and-bound framework to solve the nonconvex nonsmooth reformulation of MILPs!

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Thanks for your attention!