Sequential penalty methods for mixed integer programs

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The mixed-binary linear complementarity problem (MILCP) is the task to find a vector $z \in \mathbb{R}^n$ that satisfies

$$z \geq 0$$
$$q + Mz \geq 0$$
$$z^\top (q + Mz) = 0$$

$z_i \in \{0, 1\}$ for $i \in I \subseteq \{1, \ldots, n\}$

or to show that no such vector exists, for given

- $M \in \mathbb{R}^{n \times n}$, $M \succeq 0$
- $q \in \mathbb{R}^n$
Linear Complementarity Problems (LCPs) are an important tool for the modeling and analysis of equilibrium problems in economics, mechanics, ... [Cottle, Pang, Stone; “The Linear Complementarity Problem”; 2009]
[ Gabriel, Conejo, Fuller, Hobbs; “Complementarity modeling in energy markets”; 2012]

When a subset of variables is restricted to take integer values, i.e., $z_i \in \mathbb{Z}$ for a given index set $I \subseteq \{1, \ldots, n\}$ we fall in the context of MILCPs
A common tool in the analysis and resolution of a Linear Complementarity Problem (LCP) is its reformulation as Quadratic Problem (QP) [Cottle et al.;2009]:

\[
\begin{align*}
  z & \geq 0 \\
  q + Mz & \geq 0 \\
  z^T(q + Mz) & = 0
\end{align*}
\]

\[
\begin{align*}
  \text{min} \quad & z^T(q + Mz) \\
  \text{s.t.} \quad & q + Mz \geq 0 \\
  & z \geq 0
\end{align*}
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\[\iff\]

\[
\begin{align*}
    \min & \quad z^T(q + Mz) \\
    \text{s.t.} & \quad q + Mz \geq 0 \\
    & \quad z \geq 0
\end{align*}
\]

LCP has a solution if and only if the QP has an optimal solution with objective function value zero.
MIQP reformulation of a MILCP

Equivalently we can reformulate a MILCP into a MIQP:

\[ z \geq 0 \]
\[ q + Mz \geq 0 \]
\[ z^T(q + Mz) = 0 \]
\[ z_i \in \{0, 1\}, \ i \in I \]

\[ \begin{array}{c}
\text{min} & z^T(q + Mz) \\
\text{s.t.} & q + Mz \geq 0 \\
& z \geq 0 \\
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\end{array} \]

MILCP has a solution if and only if the MIQP has an optimal solution with objective function value zero.
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\end{align*}
\]

MILCP has a solution if and only if the MIQP has an optimal solution with objective function value zero

However, the existence of a solution of the MILCP cannot be expected in general...
...look for “approximate feasible solutions”

For practically relevant instances where non-existence occurs, one is interested in “approximate feasible solutions”:

points that minimize a certain infeasibility measure that combines both the violation of integrality conditions as well as of complementarity constraints
Penalizing the violation of complementarity and integrality

\[
\begin{align*}
\min & \quad \alpha P_C(z) + (1 - \alpha) P_I(z) \\
\text{s.t.} & \quad q + Mz \geq 0 \\
& \quad z \geq 0 \\
& \quad z_i \leq 1, \quad i \in I
\end{align*}
\]

where

- \( \alpha \in [0, 1] \)
- \( P_C(z) \) is a function penalizing the violation of the complementarity constraints
- \( P_I(z) \) is a function penalizing the violation of the integrality constraints

[Raghavachari;1969], [Giannessi, Tardella; 1998], [Zhu; 2003], [Lucidi, Rinaldi; 2010], [De Santis, Lucidi, Rinaldi; 2013]
A nonconvex, nonsmooth reformulation of MILCP

\[
\begin{align*}
\min & \quad \alpha z^\top (q + Mz) + (1 - \alpha) \sum_{i \in I} \min \{z_i, 1 - z_i\} \\
\text{s.t.} & \quad q + Mz \geq 0 \\
& \quad z \geq 0 \\
& \quad z_i \leq 1, \quad i \in I
\end{align*}
\]

\((NC_{ref})\)

where

- \(\alpha \in [0, 1]\)
- \(P_C(z) = z^\top (q + Mz)\)
- \(P_I(z) = \sum_{i \in I} \min \{z_i, 1 - z_i\}\)

\(P_I(z)\) is concave and piecewise linear
Features of the *penalty* branch-and-bound method

In order to globally solve problem NC$_{ref}$, we address a sequence of convex quadratic smooth problems that
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In order to globally solve problem $NC_{ref}$, we address a **sequence of convex quadratic smooth problems** that

- share the same feasible set
Features of the *penalty* branch-and-bound method

In order to globally solve problem $\text{NC}_{\text{ref}}$, we address a sequence of convex quadratic smooth problems that

- share the same feasible set
- progressively increase the penalization of the integrality constraint violation
Features of the *penalty* branch-and-bound method

In order to globally solve problem $NC_{\text{ref}}$, we address a sequence of convex quadratic smooth problems that

- share the same feasible set
- progressively increase the penalization of the integrality constraint violation

\[ \Downarrow \]
the objective function slightly changes along the iterations!
Problem at the root node

At the **root node** of the branch-and-bound tree, we solve the convex smooth problem

\[
\begin{align*}
\min & \quad \alpha z^T (q + Mz) \\
\text{s.t.} & \quad q + Mz \geq 0 \\
& \quad z \geq 0 \\
& \quad z_i \leq 1, \quad i \in I
\end{align*}
\]

obtained from Problem \((\text{NC}_{\text{ref}})\) by **neglecting the second term in the objective function**
Branching

Let $z^*$ be the solution of the root node relaxation
Branching

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Choose an index $j \in I$ such that
$$\min\{z_j^*, 1 - z_j^*\} > 0$$

and build two children nodes:
Branching

Let $z^*$ be the solution of the root node relaxation

Choose an index $j \in I$ such that $\min\{z_j^*, 1 - z_j^*\} > 0$ and build two children nodes:

$$\min\{z_j^*, 1 - z_j^*\} > 0$$

$$\text{add } (1 - \alpha)z_j$$

$$\text{add } (1 - \alpha)(1 - z_j)$$
Branching

Let $z^*$ be the solution of the root node relaxation

Choose an index $j \in I$ such that $\min\{z^*_j, 1 - z^*_j\} > 0$ and build two children nodes:

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\min\{z^*_j, 1 - z^*_j\} > 0
\]

\[
\text{add } (1 - \alpha)z_j \\
\text{add } (1 - \alpha)(1 - z_j)
\]

\[
\min \alpha z^T(q + Mz) + (1 - \alpha)z
\]

s.t. \[q + Mz \geq 0\]

\[
z \geq 0, \quad z_i \leq 1, \quad i \in I
\]

\[\text{aims to drive } z_j \text{ to 0 in the respective subtree}\]
Branching

Let $z^*$ be the solution of the root node relaxation

Choose an index $j \in I$ such that $\min\{z_j^*, 1 - z_j^*\} > 0$ and build two children nodes:

$$\min\{z_j^*, 1 - z_j^*\} > 0$$

- add $(1 - \alpha)z_j$
- add $(1 - \alpha)(1 - z_j)$

\[
\begin{align*}
\min \quad & \alpha z^T(q + Mz) + (1 - \alpha)z_j \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \geq 0, \quad z_i \leq 1, \quad i \in I
\end{align*}
\]

--→ aims to drive $z_j$ to 0 in the respective subtree

\[
\begin{align*}
\min \quad & \alpha z^T(q + Mz) + (1 - \alpha)(1 - z_j) \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \geq 0, \quad z_i \leq 1, \quad i \in I
\end{align*}
\]

--→ aims to drive $z_j$ to 1 in the respective subtree
Problem at the node $N = (l_0, l_1)$

A node $N = (l_0, l_1)$ is identified by two sets of indices:

- $I_0$: set of indices $j \in I$ for which $(1 - \alpha)z_j$ is added
- $I_1$: set of indices $j \in I$ for which $(1 - \alpha)(1 - z_j)$ is added

The subproblem at node $N = (I_0, I_1)$ is

\[
\min f_N(z) \quad \text{s.t.} \quad q + Mz \geq 0, \quad z \geq 0, \quad z_i \leq 1, \quad i \in I
\]

with

\[
f_N(z) = \alpha z^\top (q + Mz) + (1 - \alpha) \left( \sum_{j \in I_0} z_j + \sum_{j \in I_1} (1 - z_j) \right)
\]
Problem at the node $N = (l_0, l_1)$

A node $N = (l_0, l_1)$ is identified by two sets of indices:

- $l_0$: set of indices $j \in l$ for which $(1 - \alpha)z_j$ is added
- $l_1$: set of indices $j \in l$ for which $(1 - \alpha)(1 - z_j)$ is added

The subproblem at node $N = (l_0, l_1)$ is

$$\min f_N(z)$$

s.t.

$$q + Mz \geq 0$$
$$z \geq 0$$
$$z_i \leq 1, \quad i \in I$$

with

$$f_N(z) = \alpha z^\top (q + Mz) + (1 - \alpha) \left( \sum_{j \in I_0} z_j + \sum_{j \in I_1} (1 - z_j) \right)$$
Problem at the node $N = (l_0, l_1)$

A node $N = (l_0, l_1)$ is identified by two sets of indices:

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\min & \quad f_N(z) \\
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\end{align*}
$$

with

$$
f_N(z) = \alpha z^\top (q + Mz) + (1 - \alpha) \left( \sum_{j \in l_0} z_j + \sum_{j \in l_1} (1 - z_j) \right)
$$
Enumerating the partitions \((I_0, I_1)\) of \(I \subseteq \{1, \ldots, n\}\)

The minimum among the optimal solutions of the problems of \textbf{all leaf nodes} of the fully enumerated branch-and-bound tree is the optimal solution of Problem \((NC_{\text{ref}})\):

\begin{lemma}
Let \(z^*\) be an optimal solution of Problem \((NC_{\text{ref}})\) and \(z^*_N\) the solution at the node \(N = (I_0, I_1)\). Then,

\[ f(z^*) = \min \{ f_N(z^*_N) : N = (I_0, I_1) \text{ with } I_0 \cup I_1 = I \text{ and } I_0 \cap I_1 = \emptyset \} \]
\end{lemma}
Bounding and Pruning

The optimal value $f_{N}(z_{N}^{*})$ of the problem defined at a node $N$ is a local lower bound for the subtree rooted in $N$:

**Lemma**

Let $N' = (l'_0, l'_1)$ be a successor of $N = (l_0, l_1)$, i.e., $l_0 \subseteq l'_0$ and $l_1 \subseteq l'_1$. Then,

$$f_{N}(z_{N}^{*}) \leq f_{N'}(z_{N'}^{*})$$
Bounding and Pruning

The optimal value $f_N(z_N^*)$ of the problem defined at a node $N$ is a local lower bound for the subtree rooted in $N$:

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Let $N' = (I'_0, I'_1)$ be a successor of $N = (I_0, I_1)$, i.e., $I_0 \subseteq I'_0$ and $I_1 \subseteq I'_1$. Then,

$$f_N(z_N^*) \leq f_{N'}(z_{N'}^*)$$

$$\Downarrow$$

If $z_N^*$ is such that $f_N(z_N^*) \geq f(z_{inc}^*)$

every leaf of the subtree rooted in $N$ cannot yield a better solution than the best known solution $z_{inc}^*$
The optimal value $f_N(z_N^*)$ of the problem defined at a node $N$ is a local lower bound for the subtree rooted in $N$:

**Lemma**

Let $N' = (l'_0, l'_1)$ be a successor of $N = (l_0, l_1)$, i.e., $l_0 \subseteq l'_0$ and $l_1 \subseteq l'_1$. Then,

$$f_N(z_N^*) \leq f_{N'}(z_{N'}^*)$$

**↓**

If $z_N^*$ is such that $f_N(z_N^*) \geq f(z_{inc}^*)$

every leaf of the subtree rooted in $N$ cannot yield a better solution than the best known solution $z_{inc}^*$

and **we can prune the subtree rooted in $N$**
MILCP–PBB Scheme

**Input:** \( q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}, I \subseteq \{1, \ldots, n\}, \alpha \in (0, 1) \)

**Output:** A global optimum \( z^* \) of Problem (NC<sub>ref</sub>)

Set \( N \leftarrow \{(\emptyset, \emptyset)\}, f_{\text{inc}} \leftarrow \infty, z_{\text{inc}}^* \leftarrow \text{none} \)

while \( N \neq \emptyset \) do

Choose \( N = (I_0, I_1) \in N \)

Set \( N \leftarrow N \setminus \{N\} \)

Compute \( z_{\text{N}}^* \in \arg\min \{f_N(z) : q + Mz \geq 0, z \in [0, 1]^n\} \)

if \( f(z_{\text{N}}^*) < f_{\text{inc}} \) then

Set \( z_{\text{inc}}^* \leftarrow z_{\text{N}}^*, f_{\text{inc}} \leftarrow f(z_{\text{N}}^*) \)

end if

if \( f_N(z_{\text{N}}^*) < f_{\text{inc}} \) and \( I \setminus (I_0 \cup I_1) \neq \emptyset \) then

Choose \( j \in I \setminus (I_0 \cup I_1) \)

Set \( N \leftarrow N \cup \{(I_0 \cup \{j\}, I_1), (I_0, I_1 \cup \{j\})\} \)

end if

end while

return \( z_{\text{inc}}^* \)
MILCP–PBB Scheme

**Input:** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$

**Output:** A global optimum $z^*$ of Problem (NC\textsubscript{ref})

Set $\mathcal{N} \leftarrow \{()\}$, $f_{\text{inc}} \leftarrow \infty$, $z^*_{\text{inc}} \leftarrow \text{none}$
MILCP–PBB Scheme

**Input:** \( q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}, I \subseteq \{1, \ldots, n\}, \alpha \in (0, 1) \)

**Output:** A global optimum \( z^* \) of Problem \((\text{NC}_{\text{ref}})\)

Set \( \mathcal{N} \leftarrow \{(\emptyset, \emptyset)\}, f_{\text{inc}} \leftarrow \infty, z_{\text{inc}} \leftarrow \text{none} \)

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MILCP–PBB Scheme

Input: \( q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}, I \subseteq \{1, \ldots, n\}, \alpha \in (0, 1) \)

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Set \( \mathcal{N} \leftarrow \{(\emptyset, \emptyset)\} \), \( f_{\text{inc}} \leftarrow \infty \), \( z_{\text{inc}} \leftarrow \text{none} \)

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  Choose \( N = (I_0, I_1) \in \mathcal{N} \)
MILCP–PBB Scheme

**Input:** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$

**Output:** A global optimum $z^*$ of Problem (NC$_{ref}$)

Set $\mathcal{N} \leftarrow \{(\emptyset, \emptyset)\}$, $f_{inc} \leftarrow \infty$, $z_{inc}^* \leftarrow \text{none}$

while $\mathcal{N} \neq \emptyset$ do

Choose $N = (I_0, I_1) \in \mathcal{N}$

Set $\mathcal{N} \leftarrow \mathcal{N} \setminus \{N\}$

Compute $z_N^* \in \text{argmin}\{f_N(z) : q + Mz \geq 0, z \in [0, 1]^n\}$

if $f(N)(z_N^*) < f_{inc}$ then

Set $z_{inc}^* \leftarrow z_N^*$, $f_{inc} \leftarrow f(N)(z_N^*)$

end if

if $f_N(z_N^*) < f_{inc}$ and $I \setminus (I_0 \cup I_1) \neq \emptyset$ then

Choose $j \in I \setminus (I_0 \cup I_1)$

Set $\mathcal{N} \leftarrow \mathcal{N} \cup \{(I_0 \cup \{j\}, I_1), (I_0, I_1 \cup \{j\})\}$

end if

end while

return $z_{inc}^*$
MILCP–PBB Scheme

**Input:** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$

**Output:** A global optimum $z^*$ of Problem (NC_{ref})

Set $\mathcal{N} \leftarrow \{ (\emptyset, \emptyset) \}$, $f_{\text{inc}} \leftarrow \infty$, $z_{\text{inc}}^* \leftarrow \text{none}$

**while** $\mathcal{N} \neq \emptyset$ **do**

Choose $N = (I_0, I_1) \in \mathcal{N}$

Set $\mathcal{N} \leftarrow \mathcal{N} \setminus \{N\}$

Compute $z_N^* \in \arg \min \{ f_N(z) : q + Mz \geq 0, z \in [0, 1]^n \}$

**if** $f(z_N^*) < f_{\text{inc}}$ **then**

Set $z_{\text{inc}}^* \leftarrow z_N^*$, $f_{\text{inc}} \leftarrow f(z_N^*)$

**end if**

**end while**

return $z_{\text{inc}}^*$
MILCP–PBB Scheme

Input: \( q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}, I \subseteq \{1, \ldots, n\}, \alpha \in (0, 1) \)

Output: A global optimum \( z^* \) of Problem \((\text{NC}_{\text{ref}})\)

Set \( \mathcal{N} \leftarrow \{ (\emptyset, \emptyset) \} \), \( f_{\text{inc}} \leftarrow \infty \), \( z_{\text{inc}} \leftarrow \text{none} \)

while \( \mathcal{N} \neq \emptyset \) do

Choose \( N = (I_0, I_1) \in \mathcal{N} \)

Set \( \mathcal{N} \leftarrow \mathcal{N} \setminus \{N\} \)

Compute \( z^*_N \in \text{argmin} \{ f_N(z) : q + Mz \geq 0, z \in [0, 1]^n \} \)

if \( f(z^*_N) < f_{\text{inc}} \) then

Set \( z_{\text{inc}} \leftarrow z^*_N \), \( f_{\text{inc}} \leftarrow f(z^*_N) \)

end if

if \( f_N(z^*_N) < f_{\text{inc}} \) and \( I \setminus (I_0 \cup I_1) \neq \emptyset \) then
MILCP–PBB Scheme

**Input:** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$

**Output:** A global optimum $z^*$ of Problem (NC$_{\text{ref}}$)

Set $\mathcal{N} \leftarrow \{ (\emptyset, \emptyset) \}$, $f_{\text{inc}} \leftarrow \infty$, $z_{\text{inc}} \leftarrow \text{none}$

**while** $\mathcal{N} \neq \emptyset$ **do**

Choose $N = (I_0, I_1) \in \mathcal{N}$

Set $\mathcal{N} \leftarrow \mathcal{N} \setminus \{N\}$

Compute $z^*_N \in \text{argmin}\{f_N(z) : q + Mz \geq 0, \ z \in [0, 1]^n\}$

**if** $f(z^*_N) < f_{\text{inc}}$ **then**

Set $z_{\text{inc}} \leftarrow z^*_N$, $f_{\text{inc}} \leftarrow f(z^*_N)$

**end if**

**if** $f_N(z^*_N) < f_{\text{inc}} \text{ and } I \setminus (I_0 \cup I_1) \neq \emptyset$ **then**

Choose $j \in I \setminus (I_0 \cup I_1)$

Set $\mathcal{N} \leftarrow \mathcal{N} \cup \{ (I_0 \cup \{j\}, I_1), (I_0, I_1 \cup \{j\}) \}$

**end if**

**return** $z_{\text{inc}}$
**MILCP–PBB Scheme**

**Input:** \( q \in \mathbb{R}^n, \ M \in \mathbb{R}^{n \times n}, \ I \subseteq \{1, \ldots, n\}, \ \alpha \in (0, 1) \)

**Output:** A global optimum \( z^* \) of Problem (NC_{ref})

Set \( \mathcal{N} \leftarrow \{ (\emptyset, \emptyset) \} \), \( f_{\text{inc}} \leftarrow \infty \), \( z_{\text{inc}} \leftarrow \text{none} \)

while \( \mathcal{N} \neq \emptyset \) do

Choose \( N = (I_0, I_1) \in \mathcal{N} \)

Set \( \mathcal{N} \leftarrow \mathcal{N} \setminus \{ N \} \)

Compute \( z_{N}^* \in \text{argmin}\{ f_N(z) : q + Mz \geq 0, \ z \in [0, 1]^n \} \)

if \( f(z_{N}^*) < f_{\text{inc}} \) then

Set \( z_{\text{inc}} \leftarrow z_{N}^*, \ f_{\text{inc}} \leftarrow f(z_{N}^*) \)

end if

if \( f_N(z_{N}^*) < f_{\text{inc}} \) and \( I \setminus (I_0 \cup I_1) \neq \emptyset \) then

Choose \( j \in I \setminus (I_0 \cup I_1) \)

Set \( \mathcal{N} \leftarrow \mathcal{N} \cup \{ (I_0 \cup \{ j \}, I_1), (I_0, I_1 \cup \{ j \}) \} \)

end if

end while

return \( z_{\text{inc}} \)
MILCP–PBB Scheme

**Input:** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, $I \subseteq \{1, \ldots, n\}$, $\alpha \in (0, 1)$

**Output:** A global optimum $z^*$ of Problem (NC$_{\text{ref}}$)

Set $\mathcal{N} \leftarrow \{() \}$, $f_{\text{inc}} \leftarrow \infty$, $z_{\text{inc}} \leftarrow \text{none}$

**while** $\mathcal{N} \neq \emptyset$ **do**

Choose $N = (I_0, I_1) \in \mathcal{N}$

Set $\mathcal{N} \leftarrow \mathcal{N} \setminus \{N\}$

Compute $z_N^* \in \arg\min \{f_N(z) : q + Mz \geq 0, z \in [0, 1]^n\}$

**if** $f(z_N^*) < f_{\text{inc}}$ **then**

Set $z_{\text{inc}} \leftarrow z_N^*$, $f_{\text{inc}} \leftarrow f(z_N^*)$

**end if**

**if** $f_N(z_N^*) < f_{\text{inc}}$ **and** $I \setminus (I_0 \cup I_1) \neq \emptyset$ **then**

Choose $j \in I \setminus (I_0 \cup I_1)$

Set $\mathcal{N} \leftarrow \mathcal{N} \cup \{(I_0 \cup \{j\}, I_1), (I_0, I_1 \cup \{j\})\}$

**end if**

**end while**

**return** $z_{\text{inc}}^*$
Finite termination

Theorem

Algorithm MILCP–PBB terminates after finitely many steps with a global optimal solution of Problem \( \text{NC}_{\text{ref}} \).
Finite termination

**Theorem**

Algorithm MILCP–PBB terminates after finitely many steps with a global optimal solution of Problem \((\text{NC}_{\text{ref}})\).

**Remark**

Note that in our branch-and-bound method, there is no direct analogy to pruning due to infeasibility.

In case at a node we find a feasible solution for the MILCP we stop the algorithm.
Adding simple cuts

Within the node subproblem we include simple bound constraints:

$$\begin{align*}
\min \quad & f_N(z) \\
\text{s.t.} \quad & q + Mz \geq 0 \\
& z \in [0, 1]^n \\
& z_j \leq 0.5 \quad \text{if } j \in I_0 \\
& z_j \geq 0.5 \quad \text{if } j \in I_1
\end{align*}$$

Lemma

Let $z^*_N$ be an optimal solution at node $N$ when simple cuts are included. Then,

$$f_N(z^*_N) = \min \left\{ f_N(z^*_N) : N = (I_0, I_1) \text{ with } I_0 \cup I_1 = I \right\}$$
Adding simple cuts

Within the node subproblem we include simple bound constraints:

\[
\begin{align*}
\min & \quad f_N(z) \\
\text{s.t.} & \quad q + Mz \geq 0 \\
& \quad z \in [0, 1]^n \\
& \quad z_j \leq 0.5 \quad \text{if } j \in l_0 \\
& \quad z_j \geq 0.5 \quad \text{if } j \in l_1
\end{align*}
\]

**Lemma**

Let \( z^*_N \) be an optimal solution at node \( N \) when simple cuts are included. Then,

\[
f(z^*) = \min \{ f_N(z^*_N) : N = (l_0, l_1) \text{ with } l_0 \cup l_1 = l \}
\]
Adding simple cuts
finite termination

Lemma
Let $N' = (I_0', I_1')$ be a successor of some node $N = (I_0, I_1)$ in the branching tree, i.e., $I_0 \subseteq I_0'$ and $I_1 \subseteq I_1'$ holds. Further, let $z_N^*, z_{N'}^*$ be optimal solutions of nodes $N$ and $N'$, respectively, when simple cuts are used. Then,

$$f_N(z_N^*) \leq f_{N'}(z_{N'}^*)$$

holds.

Theorem
Algorithm MILCP−PBB remains correct when simple cuts

$$z_j \leq 0.5 \text{ for all } j \in I_0, \quad z_j \geq 0.5 \text{ for all } j \in I_1$$

are added at any node $N = (I_0, I_1)$. 
Numerical results
Randomly generated instances

We built matrices $M \in \mathbb{R}^{n \times n}$ with $n \in \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500\}$. 

We then built vectors $q \in \mathbb{R}^n$ in four different ways, each reflecting a certain "degree of feasibility" in the resulting instance. More precisely, we built instances for which $z \in \mathbb{R}^n$ exists so that

(a) only $q + Mz \geq 0$, $z \geq 0$ are guaranteed to be satisfied,

(b) only $q + Mz \geq 0$, $z \geq 0$ and $z_i \in \{0, 1\}$, $i \in I$ are guaranteed to be satisfied,

(c) only $q + Mz \geq 0$, $z \geq 0$ and complementarity $(z^* \top (q + Mz^*)) = 0)$ are guaranteed to be satisfied.

We created 10 instances for every size $n$ and the types (a)–(c), yielding 300 different instances in total.
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(c) only $q + Mz \geq 0$, $z \geq 0$ and complementarity $(z^\star \top (q + Mz^\star) = 0)$ are guaranteed to be satisfied,

We created 10 instances for every size $n$ and the types (a)–(c), yielding 300 different instances in total.
Numerical results on the use of simple cuts

Performance Profiles

Figure: Performance profiles: number of nodes (left), running time (right)
Numerical comparison on branching rules

Performance Profiles

Figure: Performance profiles: number of nodes (left), running time (right)
MIQP-based branching rule

We presolve single-binary-variable MIQPs, one for each $z_j$, $j \in I$:

\[
\begin{align*}
\min_{z \in \mathbb{R}^n} & \quad z^T (q + Mz) \\
\text{s.t.} & \quad q + Mz \geq 0, \; z \geq 0, \\
& \quad z_j \in \{0, 1\}.
\end{align*}
\]

measuring the impact of the $j$-th variable on the infeasibility of the problem
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measuring the impact of the $j$-th variable on the infeasibility of the problem

We sort the indices $j \in I$ in decreasing order with respect to the optimal objective function values
Comparison with other approaches
An MILP reformulation, with additional binary variables and big-M constraints

[Gabriel, Conejo, Ruiz, Siddiqui; 2013]

\[
\begin{align*}
\min_{z,z',z'',\rho,\sigma} & \quad \alpha \sum_{i=1}^{n} \rho_i + (1 - \alpha) \sum_{i \in I} \sigma_i \\
\text{s.t.} & \quad z \geq 0, \quad q + Mz \geq 0, \quad (2a) \\
& \quad z \leq Bz' + \rho, \quad (2b) \\
& \quad q + Mz \leq B(1 - z') + \rho, \quad (2c) \\
& \quad 0 \leq z_l \leq z'' + \sigma, \quad (2d) \\
& \quad z'' - \sigma \leq z_l \leq 1, \quad (2e) \\
& \quad z \in \mathbb{R}^n, \quad z' \in \{0,1\}^n, \quad z'' \in \{0,1\}^I, \quad (2f) \\
& \quad \sigma \in \mathbb{R}_{\geq 0}^I, \quad \rho \in \mathbb{R}_{\geq 0}^n. \quad (2g) \\
\end{align*}
\]

# variables: \(3n + 2|I|, \quad (n + |I| \text{ constrained to be binary})\)
Comparison with other approaches

An MIQP reformulation, no big-M constraints

\[
\begin{align*}
\min_{z, z', \sigma} & \quad \alpha z^\top (q + Mz) + (1 - \alpha) \sum_{i \in I} \sigma_i \\
\text{s.t.} & \quad z \geq 0, \quad q + Mz \geq 0, \\
& \quad 0 \leq z_I \leq z' + \sigma, \\
& \quad z' - \sigma \leq z_I \leq 1, \\
& \quad z \in \mathbb{R}^n, \quad z' \in \{0, 1\}^l, \\
& \quad \sigma \in \mathbb{R}_{\geq 0}^l.
\end{align*}
\]

\# variables: $n + 2|I|$, ($|I|$ constrained to be binary)
Comparison with Gurobi addressing the MILP and the MIQP reformulations

Figure: Performance profiles: number of nodes (left), running time (right)
Comparison with GUROBI addressing the MIQP

Harder test set (300 instances with $n = 100, \ldots, 600$)

Figure: Performance profiles: number of nodes (left), running time (right)
Conclusions

We presented a **penalty branch-and-bound method** for MILCPs

- the method is able to compute a solution *if one exists* or it computes an approximate solution that minimizes an infeasibility measure based on the violation of the integrality and complementarity conditions of the problem

- the objective function slightly changes along the nodes so that the penalization of the integrality constraint violation is progressively increased
Future work

...useful for MILPs?

Under specific assumption on $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ we can prove that $\epsilon > 0$ exists such that

$$\min c^\top x \quad \iff \quad \min c^\top x + \frac{1}{\epsilon} \sum_{i \in I} \min \{x_i, 1 - x_i\}$$

s.t. $Ax \leq b$

$x_i \in \{0, 1\}, \quad i \in I$

s.t. $Ax \leq b$

$x \in [0, 1]^n$
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$x_i \in \{0, 1\}, \quad i \in I$

we can use our branch-and-bound framework to solve the nonconvex nonsmooth reformulation of MILPs!
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\[
\begin{align*}
\min & \quad c^\top x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x_i \in \{0, 1\}, \quad i \in I
\end{align*}
\]

\[
\begin{align*}
\leftrightarrow
\min & \quad c^\top x + \frac{1}{\epsilon} \sum_{i \in I} \min\{x_i, 1 - x_i\} \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \in [0, 1]^n
\end{align*}
\]

we can use our branch-and-bound framework to solve the nonconvex nonsmooth reformulation of MILPs!

Thanks for your attention!