Dyadic Linear Programming

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Dyadic Linear Programming

A rational number is dyadic if it is an integer multiple of $\frac{1}{2^k}$ for some nonnegative integer k.

Dyadic numbers can be represented exactly on a computer in binary floating-point arithmetic.
Therefore they are important for numerical computations.

$$DLP \qquad \begin{array}{c} \sup w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}$$

where a vector x is dyadic if each of its components is a dyadic rational.

If we require dyadic numbers with k bounded by a fixed given K, (DLP) reduces to integer programming.

Our Initial Motivation

For a 0,1 matrix A, consider the primal-dual pair

$$\min\left\{w^\top x : Ax \geq \mathbf{1}, x \geq 0\right\} \quad = \quad \max\left\{\mathbf{1}^\top y : A^\top y \leq w, y \geq 0\right\}.$$

THE DYADIC CONJECTURE Seymour 1975
If the primal has an integral optimal solution for every $w \in \mathbb{Z}_+^n$, then the dual has an optimal solution that is dyadic for every $w \in \mathbb{Z}_+^n$.

THEOREM Abdi, Cornuéjols, Guenin, Tuncel (SIDMA 2022)
The dyadic conjecture is true if the optimal value is 2.

THEOREM Abdi, Cornuéjols, Palion 2022
The dyadic conjecture is true for T-joins.

Actually, for T-joins, Seymour conjectured a $1/4$-integral solution.

Some questions

$$DLP \qquad \begin{array}{c} \sup w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}$$

- When is this problem feasible ?
- When does it have an optimum solution ?
- Can feasibility be checked in polynomial time ?
- Can dyadic linear programs be solved in polynomial time ?
- When can we guarantee that the dual also has a dyadic optimal solution ?

Infeasibility certificate ?

A dyadic linear program may have no optimal solution :
sup x : $3x \leq 1$, x dyadic.

Or it might not be feasible :
sup x : $3x = 1$, x dyadic.

When a dyadic linear program is infeasible, can one provide a certificate of infeasibility ?

We answer this question in the affirmative.
We show that deciding feasibility of a dyadic linear program belongs to $\mathrm{NP} \cap \mathrm{coNP}$, foreshadowing a later result that the problem, in fact, belongs to P.

Feasibility

LEMMA

Consider a linear system $Ax = b$, where A, b have integral entries. Then exactly one of the following statements holds :

1. $Ax = b$ has a dyadic solution,
2. there exists a vector $y \in \mathbb{R}^m$ such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

Compare to the Integer Farkas Lemma :

Consider a linear system $Ax = b$, where A, b have integral entries. Then exactly one of the following statements holds :

1. $Ax = b$ has an integral solution,
2. there exists a vector $y \in \mathbb{R}^m$ such that $y^\top A$ is integral and $y^\top b$ is not integer.

Dyadic solutions to linear programs

LEMMA

A nonempty rational polyhedron contains a dyadic point if and only if its affine hull contains a dyadic point.

THEOREM

Let P be a nonempty rational polyhedron whose affine hull is $\{x : Ax = b\}$.

Exactly one of the following statements holds.

- P contains a dyadic point,
- there exists a y such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

Application : Cycle double covers

Let $G = (V, E)$ be a graph.
Let A be the $0, 1$ matrix whose rows correspond to E and whose columns are the incidence vectors of the cycles of G.

THE CYCLE DOUBLE COVER CONJECTURE
Szekeres 1973, Seymour 1981
$Ax = \mathbf{1}, x \geq 0$ has a $1/2$-integral solution.

THEOREM Goddyn 1993
$Ax = \mathbf{1}$ has a $1/2$-integral solution.

COROLLARY
$Ax = \mathbf{1}, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator ?

Another application : Perfect matchings

Let $G = (V, E)$ be an r-graph. That is G is an r-regular graph on an even number of vertices, and $|\delta(U)| \geq r$ for all odd cardinality $U \subseteq V$.
Let A be the $0, 1$ matrix whose rows correspond to E and whose columns are the incidence vectors of the perfect matchings of G.

THE GENERALIZED BERGE-FULKERSON CONJECTURE
Seymour 1979
$Ax = \mathbf{1}, x \geq 0$ has a $1/2$-integral solution.

THEOREM   Seymour 1979 $r = 3$,   Lovász 1987 $r \geq 4$
$Ax = \mathbf{1}$ has a $1/2$-integral solution.

COROLLARY
$Ax = \mathbf{1}, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator ?

Optimization

$$
DLP \qquad \begin{array}{c} \sup\, w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}
$$

THEOREM The status of (DLP) can be classified as follows :

1. (DLP) is infeasible,
2. (DLP) is unbounded,
3. (DLP) has an optimal solution,
4. (DLP) is not unbounded, has feasible solution(s) and a finite optimal value, but no optimal solution.

Moreover, in cases 3 and 4, the value of the supremum in (DLP) is the max objective value of the underlying LP.

Complexity

LEMMA
The feasibility problem for dyadic linear programs can be solved in polynomial time.

THEOREM
Dyadic linear programs can be solved in polynomial time.

Totally dual dyadic systems

DEFINITION

$Ax \leq b$ is **totally dual dyadic** if, for all $w \in \mathbb{Z}^n$ for which $\min\left\{b^\top y : A^\top y = w, y \geq 0\right\}$ has a solution, it has a dyadic optimal solution.

THEOREM The following are equivalent :

- $Ax \leq b$ is totally dual dyadic.
- For every nonempty face F the tight rows of A form a dyadic generating set for the conic hull. (the dyadic analogue of a Hilbert basis)
- For every nonempty face F the tight rows of A form a dyadic generating set for the span.

Dyadic generating sets for subspaces

DEFINITION $\{a^1, \ldots, a^m\}$ is a dyadic generating set for the span if every integral vector in the span of these m vectors can be expressed as a dyadic linear combination of $\{a^1, \ldots, a^m\}$.

THEOREM Let A be an integral matrix. The following are equivalent :

- the rows of A form a dyadic generating set for the span.
- the columns of A form a dyadic generating set for the span.
- whenever $y^\top A$ and Ax are integral, $y^\top Ax$ is dyadic.
- every elementary divisor of A is a power of 2.
- the GCD of the subdeterminants of A of order $\text{rank}(A)$ is a power of 2.
- there exists a dyadic matrix B such that $ABA = A$

Totally dual dyadic systems

THEOREM One can check in polynomial time whether the rows of an integral matrix form a dyadic generating set for the span.

COROLLARY
Testing total dual dyadicness belongs to coNP.

Question
What is the complexity of testing total dual dyadicness ?

COROLLARY
If every subdeterminant of A is 0 or $\pm$ a power of 2 , then $Ax \leq b$ is totally dual dyadic.

Future work

- Computer implementation of dyadic linear programming : Fast implementation of an infeasibility certificate. Fast implementation of an optimization algorithm when a dyadic optimum solution exists. Approximation using dyadic numbers.

- What about "dyadic convex programming" ?
  Our proof of the "Affine Hull Lemma" shows more generally

  Lemma A nonempty convex set whose affine hull is rational contains a dyadic point if, and only if, its affine hull contains a dyadic point.