Dyadic Linear Programming

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Dyadic Linear Programming

A rational number is dyadic if it is an integer multiple of $\frac{1}{2^k}$ for some nonnegative integer $k$.

Dyadic numbers can be represented exactly on a computer in binary floating-point arithmetic. Therefore they are important for numerical computations.

\[
\begin{align*}
\text{DLP} & \quad \sup w^\top x \\
& \quad Ax \leq b \\
& \quad x \text{ dyadic}
\end{align*}
\]

where a vector $x$ is dyadic if each of its components is a dyadic rational.

If we require dyadic numbers with $k$ bounded by a fixed given $K$, (DLP) reduces to integer programming.
Our Initial Motivation

For a 0,1 matrix $A$, consider the primal-dual pair

$$\min \left\{ w^\top x : Ax \geq 1, x \geq 0 \right\} = \max \left\{ 1^\top y : A^\top y \leq w, y \geq 0 \right\}.$$ 

**THE DYADIC CONJECTURE** Seymour 1975

If the primal has an integral optimal solution for every $w \in \mathbb{Z}^n_+$, then the dual has an optimal solution that is dyadic for every $w \in \mathbb{Z}^n_+$.

**THEOREM** Abdi, Cornuéjols, Guenin, Tuncel (SIDMA 2022)

The dyadic conjecture is true if the optimal value is 2.

**THEOREM** Abdi, Cornuéjols, Palion 2022

The dyadic conjecture is true for $T$-joins.

Actually, for $T$-joins, Seymour conjectured a $1/4$-integral solution.
Some questions

\[ DLP \]
\[
\sup w^T x \\
Ax \leq b \\
x \text{ dyadic}
\]

- When is this problem feasible?
- When does it have an optimum solution?
- Can feasibility be checked in polynomial time?
- Can dyadic linear programs be solved in polynomial time?
- When can we guarantee that the dual also has a dyadic optimal solution?
A dyadic linear program may have no optimal solution:
\[ \sup x : 3x \leq 1, \ x \text{ dyadic}. \]

Or it might not be feasible:
\[ \sup x : 3x = 1, \ x \text{ dyadic}. \]

When a dyadic linear program is infeasible, can one provide a certificate of infeasibility?

We answer this question in the affirmative. We show that deciding feasibility of a dyadic linear program belongs to \( \text{NP} \cap \text{coNP} \), foreshadowing a later result that the problem, in fact, belongs to \( \mathbb{P} \).
Feasibility

**LEMMA**
Consider a linear system $Ax = b$, where $A, b$ have integral entries. Then exactly one of the following statements holds:

1. $Ax = b$ has a dyadic solution,
2. there exists a vector $y \in \mathbb{R}^m$ such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

Compare to the **Integer Farkas Lemma**:

Consider a linear system $Ax = b$, where $A, b$ have integral entries. Then exactly one of the following statements holds:

1. $Ax = b$ has an integral solution,
2. there exists a vector $y \in \mathbb{R}^m$ such that $y^\top A$ is integral and $y^\top b$ is not integer.
LEMMA
A nonempty rational polyhedron contains a dyadic point if and only if its affine hull contains a dyadic point.

THEOREM
Let $P$ be a nonempty rational polyhedron whose affine hull is $\{x : Ax = b\}$.
Exactly one of the following statements holds.

- $P$ contains a dyadic point,
- there exists a $y$ such that $y^T A$ is integral and $y^T b$ is non-dyadic.
Application : Cycle double covers

Let $G = (V, E)$ be a graph.
Let $A$ be the $0, 1$ matrix whose rows correspond to $E$ and whose columns are the incidence vectors of the cycles of $G$.

THE CYCLE DOUBLE COVER CONJECTURE
Szekeres 1973, Seymour 1981
$Ax = 1, x \geq 0$ has a $1/2$-integral solution.

THEOREM Goddyn 1993
$Ax = 1$ has a $1/2$-integral solution.

COROLLARY
$Ax = 1, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator ?
Another application : Perfect matchings

Let $G = (V, E)$ be an $r$-graph. That is $G$ is an $r$-regular graph on an even number of vertices, and $|\delta(U)| \geq r$ for all odd cardinality $U \subseteq V$.

Let $A$ be the $0, 1$ matrix whose rows correspond to $E$ and whose columns are the incidence vectors of the perfect matchings of $G$.

THE GENERALIZED BERGE-FULKERSON CONJECTURE
Seymour 1979
$Ax = 1, x \geq 0$ has a $1/2$-integral solution.

THEOREM Seymour 1979 $r = 3$, Lovász 1987 $r \geq 4$
$Ax = 1$ has a $1/2$-integral solution.

COROLLARY
$Ax = 1, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator?
Optimization

\[ \sup_{x} w^T x \]

**THEOREM** The status of (DLP) can be classified as follows:

1. (DLP) is infeasible,
2. (DLP) is unbounded,
3. (DLP) has an optimal solution,
4. (DLP) is not unbounded, has feasible solution(s) and a finite optimal value, but no optimal solution.

Moreover, in cases 3 and 4, the value of the supremum in (DLP) is the max objective value of the underlying LP.
Complexity

**LEMMA**
The feasibility problem for dyadic linear programs can be solved in polynomial time.

**THEOREM**
Dyadic linear programs can be solved in polynomial time.
Totally dual dyadic systems

**DEFINITION**
$Ax \leq b$ is totally dual dyadic if, for all $w \in \mathbb{Z}^n$ for which
$$\min \left\{ b^T y : A^T y = w, y \geq 0 \right\}$$
has a solution, it has a dyadic optimal solution.

**THEOREM** The following are equivalent:

- $Ax \leq b$ is totally dual dyadic.
- For every nonempty face $F$ the tight rows of $A$ form a dyadic generating set for the conic hull. (the dyadic analogue of a Hilbert basis)
- For every nonempty face $F$ the tight rows of $A$ form a dyadic generating set for the span.
DEFINITION \( \{a^1, \ldots, a^m\} \) is a dyadic generating set for the span if every integral vector in the span of these \( m \) vectors can be expressed as a dyadic linear combination of \( \{a^1, \ldots, a^m\} \).

THEOREM Let \( A \) be an integral matrix. The following are equivalent:

\[ \begin{align*}
&\text{the rows of } A \text{ form a dyadic generating set for the span.} \\
&\text{the columns of } A \text{ form a dyadic generating set for the span.} \\
&\text{whenever } y^\top A \text{ and } Ax \text{ are integral, } y^\top Ax \text{ is dyadic.} \\
&\text{every elementary divisor of } A \text{ is a power of 2.} \\
&\text{the GCD of the subdeterminants of } A \text{ of order } \text{rank}(A) \text{ is a power of 2.} \\
&\text{there exists a dyadic matrix } B \text{ such that } ABA = A
\]
Totally dual dyadic systems

**THEOREM** One can check in polynomial time whether the rows of an integral matrix form a dyadic generating set for the span.

**COROLLARY**
Testing total dual dyadicness belongs to coNP.

**Question**
What is the complexity of testing total dual dyadicness?

**COROLLARY**
If every subdeterminant of $A$ is 0 or $\pm$ a power of 2, then $Ax \leq b$ is totally dual dyadic.
Future work

- Computer implementation of dyadic linear programming: Fast implementation of an infeasibility certificate. Fast implementation of an optimization algorithm when a dyadic optimum solution exists. Approximation using dyadic numbers.

- What about ”dyadic convex programming”?
  Our proof of the ”Affine Hull Lemma” shows more generally

  **Lemma** A nonempty convex set whose affine hull is rational contains a dyadic point if, and only if, its affine hull contains a dyadic point.