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Binary extended formulations and sequential convexification

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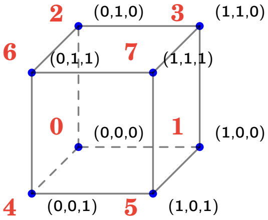
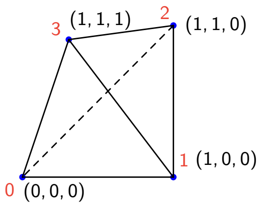


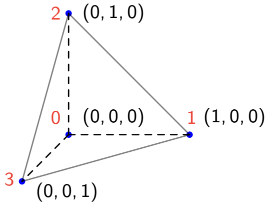
Binarizations

Let x be a variable that ranges from 0 to k.

A binarization of x is a linear formulation with variables x and $y_1, \dots, y_d$ (between 0 and 1), so that integrality of x is implied by the integrality of $y_1, \dots, y_d$.

- Unary: $x = \sum_{i=1}^{k} y_i$ with $y_1 \geq \cdots \geq y_k$ [Roy 07]
- Full: $x = \sum_{i=1}^{k} i \cdot y_i$ with $\sum_{i=1}^{k} y_i \leq 1$. [Sherali, Adams 13]





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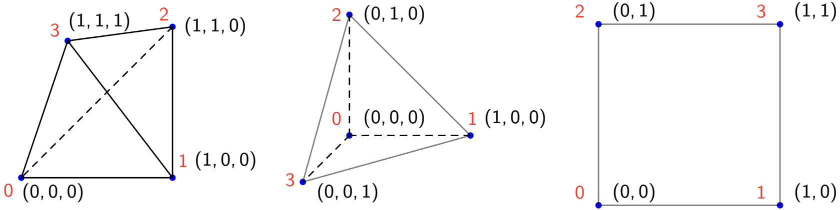
- Logarithmic: $x = \sum_{i=1}^{t} 2^{i-1} y_i$, with $k = 2^t - 1$ **[Owen, Mehrotra 02]**



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A polytope $B \subseteq \{(x, y) : (x, y) \in \mathbb{R} \times [0, 1]^d\}$ is a binarization of x in the range $\{0, \ldots, k\}$ if

$$\pi_x(\{(x, y) \in B : y \in \{0, 1\}^d\}) = \{0, \ldots, k\}.$$

Why binarizations, and which one?

IP solvers deal more easily with binary variables than general integer variables.

- Cutting planes generated from variables of a binarizations can be more effective. [Bonami, Margot 15]
- Unimodular (generalization of full and unary) are optimal in terms of split closure, but they have k variables. [Dash, Gunluk, Hildebrand 18]

But...

- The logarithmic binarization has only $O(\log k)$ variables, but can lead to worse B&B trees than original formulation. [Owen, Mehrotra 02]
- "Although this substitution is valid, it should be avoided if possible." [Optimization Modelling with LINGO]

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We propose a different way to compare binarizations inspired by a connection with sequential convexification.

Our contributions

- We propose a "natural" notion of binarizations and we characterize the vertices of formulations that use such binarizations.
- We define the rank of a binarization, related to sequential convexification and the lift-and-project rank
- We give formulas for the rank of the binarizations known in the literature, and show that
  - Unary is better than full
  - Logarithmic is optimal (among those with the same number of variables).

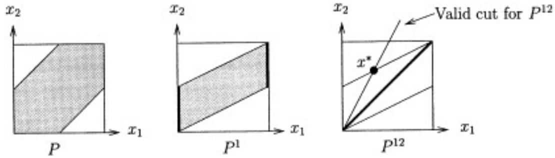
Sequential convexification

The convexification a polytope Q with respect to a binary variable x_i is

$$Q_{x_i} := \text{conv}\left(\{x \in Q : x_i = 0\} \cup \{x \in Q : x_i = 1\}\right).$$

If $Q \subset [0,1]^p \times \mathbb{R}^{n-p}$, one has

$$\text{conv}\{x \in Q : x_i \in \{0,1\}\ \forall i \in [p]\} = (((Q_{x_1})_{x_2})\dots)_{x_p}.$$



[Balas Perregaard 02]

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The lift-and-project rank of Q is the minimum integer k such that there are $i_1, \ldots, i_k \in [p]$ such that

$$\text{conv}\{x \in Q : x_i \in \{0,1\}\ \forall i \in [p]\} = ((Q_{x_{i_1}})\ldots)_{x_{i_k}}$$

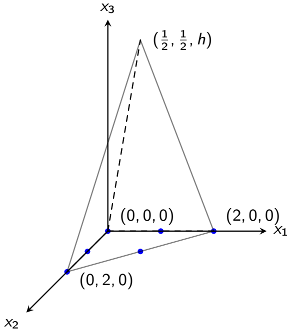
One can see this as a hitting set problem: convexifying wrt x_i we "get rid" of all vertices of Q whose x_i-component is fractional.

We need to pick $i_1, \ldots, i_k \in [p]$ so that each fractional vertex of Q has a fractional component in some of $i_1, \ldots, i_k$.

Sequential convexification converges in a finite number of steps to the integer hull, while general disjunctions do not converge.

In this example, using split disjunctions does not converge if only x_1, x_2 are required to be integer.

But, if we associate to x_1, x_2 a binarization, we obtain the integer hull by convexifying a small number of 0/1 variables.



[Cook, Kannan, Schrijver 90]

Natural binarizations and their vertices

We consider a polytope $P \subseteq [0, k]^n$ and a binary extended formulation

$$Q := \{(x, y) \in \mathbb{R}^n \times [0, 1]^{nk} : x \in P, \, (x_i, y_i) \in B_i \, \forall i \in [n]\}.$$

where B_i is a binarization for x_i.

Convexifying all the y-variables leads to the integer hull $P_I = P \cap \mathbb{Z}^n$.

In order to study the lift-and-project rank of Q, we would like to understand its vertices...

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We can characterize exactly the vertices of Q, and their x-projections, under a natural assumption.

Definition

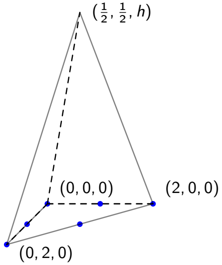
A binarization B is natural if, for each vertex (x,y) of B, x is integer.

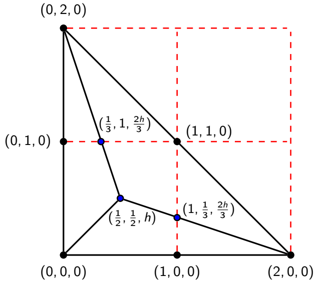
Theorem

Let $P \subseteq [0,k]^n$ be a polytope and let Q be a binary extended formulation of P with natural binarizations. Then $\bar{x} \in \mathbb{R}^n$ is a point in $\pi_x(V(Q))$ if and only if there exist $I \subseteq [n]$, $\alpha_i \in \mathbb{Z}$ for $i \in I$, and a face F of P of dimension $|I|$ such that

$$F \cap \{x_i = \alpha_i \; \forall i \in I\} = \{\bar{x}\}.$$

Projections of vertices are exactly the 0-dimensional intersections of faces of P with the integer grid.





Theorem

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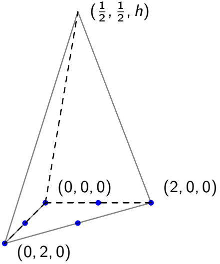
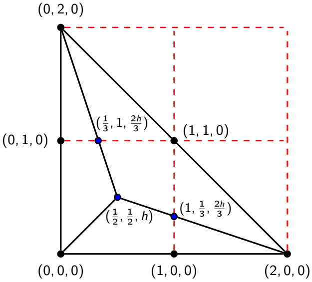
In particular, projections of vertices of Q do not depend on the binarizations used!





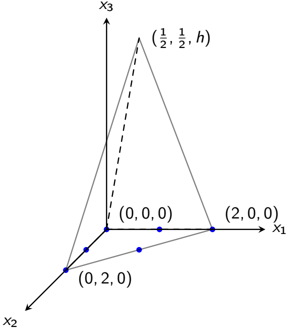
Theorem

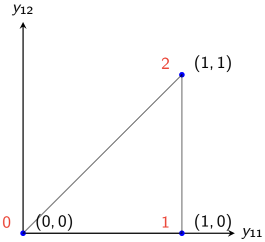
Let $P \subseteq [0, k]^n$ be a polytope and let Q be a binary extended formulation of P with natural binarizations. Then $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times [0, 1]^{nd}$ is a vertex of Q if and only if there exist $I \subseteq [n]$, $\alpha_i \in \mathbb{Z}$ for $i \in I$, and a face F of P of dimension $|I|$ such that:

- $F \cap \{x_i = \alpha_i \, \forall i \in I\} = \{\bar{x}\}$;
- $(\bar{x}_i, \bar{y}_i) \in V(B_i) \, \forall i \in I$;
- $(\bar{x}_i, \bar{y}_i) \in V(B_i \cap \{x_i = \bar{x}_i\}) \, \forall i \in [n] \setminus I$.

$$P = \{(x_1, x_2, x_3) \in [0,2]^2 \times \mathbb{R} : \quad hx_1 + hx_2 + x_3 \leq 2h$$
$$x_3 \leq 2hx_1$$
$$x_3 \leq 2hx_2$$
$$x_3 \geq 0\}$$

For $i = 1, 2,\ B_i = \{(x_i, y_{i1}, y_{i2}) \in \mathbb{R} \times [0,1]^2 : x_i = y_{i1} + y_{i2},\ y_{i1} \geq y_{i2}\}$





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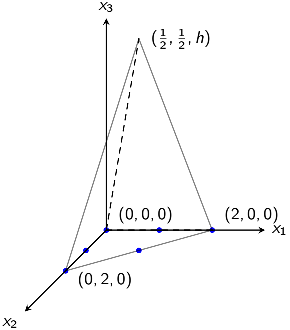
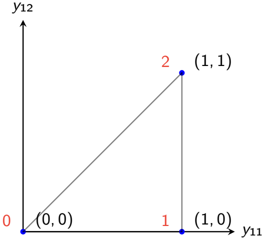
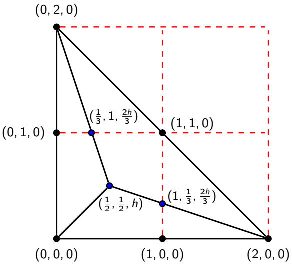
$$
\begin{aligned}
Q = \{ \quad & (x_1, x_2, x_3) \in [0,2]^2 \times \mathbb{R}, \quad (y_{11}, y_{12}, y_{21}, y_{22}) \in [0,1]^4 : \\
& hx_1 + hx_2 + x_3 \leq 2h \\
& x_3 \leq 2hx_1 \\
& x_3 \leq 2hx_2 \\
& x_3 \geq 0 \\
& (x_i, y_{i1}, y_{i2}) \in B_i \qquad\qquad i = 1, 2\}
\end{aligned}
$$





$V(Q)$ consists of the following points:

x_1	x_2	x_3	y_{11}	y_{12}	y_{21}	y_{22}
0	0	0	0	0	0	0
2	0	0	1	1	0	0
0	2	0	0	0	1	1
$1/2$	$1/2$	h	$1/2$	0	$1/2$	0
$1/2$	$1/2$	h	$1/2$	0	$1/4$	$1/4$
$1/2$	$1/2$	h	$1/4$	$1/4$	$1/2$	0
$1/2$	$1/2$	h	$1/4$	$1/4$	$1/4$	$1/4$
1	0	0	1	0	0	0
0	1	0	0	0	1	0
1	1	0	1	0	1	0
1	1	0	$1/2$	$1/2$	1	0
1	1	0	1	0	$1/2$	$1/2$
1	$1/3$	$2h/3$	1	0	$1/3$	0
1	$1/3$	$2h/3$	1	0	$1/6$	$1/6$
$1/3$	1	$2h/3$	$1/3$	0	1	0
$1/3$	1	$2h/3$	$1/6$	$1/6$	1	0



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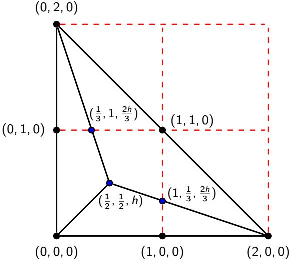
x_1	x_2	x_3	y_{11}	y_{12}	y_{21}	y_{22}
0	0	0	0	0	0	0
2	0	0	1	1	0	0
0	2	0	0	0	1	1
1/2	1/2	h	1/2	0	1/2	0
1/2	1/2	h	1/2	0	1/4	1/4
1/2	1/2	h	1/4	1/4	1/2	0
1/2	1/2	h	1/4	1/4	1/4	1/4
1	0	0	1	0	0	0
0	1	0	0	0	1	0
1	1	0	1	0	1	0
1	1	0	1/2	1/2	1	0
1	1	0	1	0	1/2	1/2
1	1/3	$2h/3$	1	0	1/3	0
1	1/3	$2h/3$	1	0	1/6	1/6
1/3	1	$2h/3$	1/3	0	1	0
1/3	1	$2h/3$	1/6	1/6	1	0



Convexifying variables y_{11}, y_{21} is enough to obtain the integer hull.

Rank of a binarization

The structure of the hitting set problem of a binary extended formulation depends on both P and the binarizations and can be complex. However, the situations simplifies if we restrict to a single x_i and B.

Let $\alpha \in \mathbb{Z}$. What is the minimum number of variables y_{ij} to convexify in order to get rid of all vertices $(x, y) \in Q$ with $\alpha < x_i < \alpha + 1$?

Thanks to our characterization of vertices of Q, it turns out that, if B is natural, the answer only depends on B and α, and not on P!

Given any binary extended formulation where natural binarization B is associated to variable x_i, and $\alpha \in \mathbb{Z}$, the rank $\mathrm{rk}_B(\alpha)$ is the minimum number of variables y_{ij} of B to convexify in order to get rid of all vertices $(x, y) \in Q$ with $\alpha < x_i < \alpha + 1$.

Intuitively, $\mathrm{rk}_B(\cdot)$ measures the progress made towards ensuring the integrality of x_i via application of sequential convexification.

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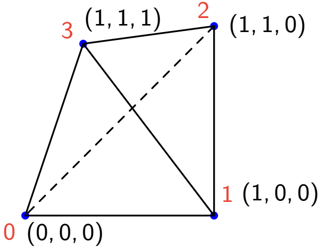
Intuitively, $\mathrm{rk}_B(\cdot)$ measures the progress made towards ensuring the integrality of x_i via application of sequential convexification.

For $\alpha_1, \dots, \alpha_\ell \in \mathbb{Z}$, the rank $\mathrm{rk}_B(\alpha_1, \dots, \alpha_\ell)$ is the minimum number of variables y_{ij} of B to convexify in order to get rid of all vertices $(x, y) \in Q$ with $\alpha_j < x_i < \alpha_j + 1$ for any $j = 1, \dots, \ell$.

$\text{rk}_B(\alpha)$ = minimum number of variables y_{ij} of B to convexify in order to get rid of all vertices $(x, y) \in Q$ with $\alpha < x_i < \alpha + 1$.

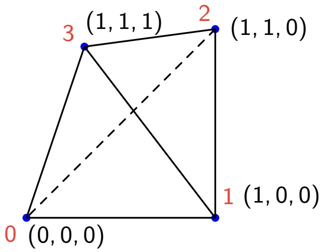
$$B = \{(x_i, y) \in \mathbb{R} \times [0,1]^3 : x_i = \sum_{j=1}^{d} y_j,\ 1 \geq y_1 \geq y_2 \geq y_3 \geq 0\};$$



$\text{rk}_B(\alpha) = 1$ for $\alpha = 0, 1, 2$

Given a natural binarization $B \subseteq [0, k] \times [0, 1]^d$ and $\alpha \in \{0, \dots, k-1\}$, we say that edge $((x^u, y^u), (x^v, y^v))$ of B is an α-edge if $x^u \leq \alpha$ and $x^v \geq \alpha + 1$, or viceversa.

The indicator set of edge $((x^u, y^u), (x^v, y^v))$ is the set of indices $i \in [d]$ for which $y_i^u \neq y_i^v$.



0-edges	sets
$0-1$	$\{1\}$
$0-2$	$\{1, 2\}$
$0-3$	$\{1, 2, 3\}$

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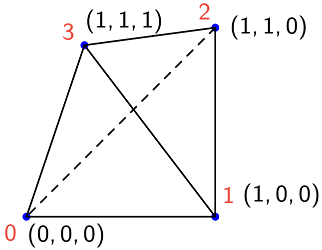
Lemma

$\text{rk}_B(\alpha_1, \dots, \alpha_\ell) = \min |I| : I \subseteq [d]$ hits the indicator sets of all α_j-edges of B, for $j \in [\ell]$.

Proof idea: the rank is also equal to the lift-and-project rank of a certain polytope inside B.

Unary binarization

$$B^U = \{(x,y) \in \mathbb{R} \times [0,1]^k : x = \textstyle\sum_{i=1}^k y_i,\ 1 \geq y_1 \geq \cdots \geq y_k \geq 0\};$$

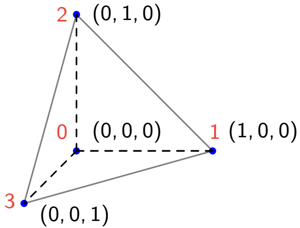


1-edges	sets
$0-2$	$\{1,2\}$
$0-3$	$\{1,2,3\}$
$1-2$	$\{2\}$
$1-3$	$\{2,3\}$

$$\mathrm{rk}_{B^U}(\alpha_1, \dots, \alpha_\ell) = \ell.$$

Full binarization

$$B^F = \{(x,y) \in \mathbb{R} \times [0,1]^k : x = \sum_{i=1}^k i \cdot y_i,\ \sum_{i=1}^k y_i \le 1\};$$



0-edges	sets
$0-1$	$\{1\}$
$0-2$	$\{2\}$
$0-3$	$\{3\}$

$$\mathrm{rk}_{B^F}(\alpha_1, \ldots, \alpha_\ell) = k - \min_{j \in [\ell]} \alpha_j.$$

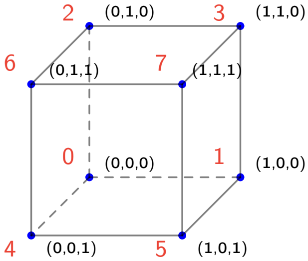
$k - \min_{j\in[\ell]} \alpha_j \ge k - (k-\ell) = \ell$, hence:

Unary has smaller rank than Full:

$$\mathrm{rk}_{B^F}(\cdots) \ge \mathrm{rk}_{B^U}(\cdots).$$

Logarithmic binarization

$$B^L = \{(x, y) \in \mathbb{R} \times [0,1]^d : x = \sum_{i=1}^{d} 2^{i-1} y_i\}.$$

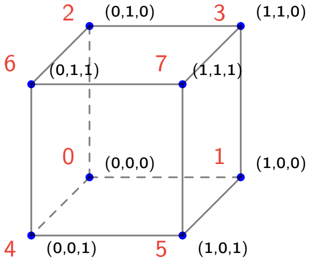


0-edges	sets
$0 - 1$	$\{1\}$
$0 - 2$	$\{2\}$
$0 - 4$	$\{3\}$

Observation: indicator sets of α-edges are singletons, and parallel edges have the same indicator set.

Logarithmic binarization

$$B^L = \{(x, y) \in \mathbb{R} \times [0,1]^d : x = \sum_{i=1}^{d} 2^{i-1} y_i\}.$$

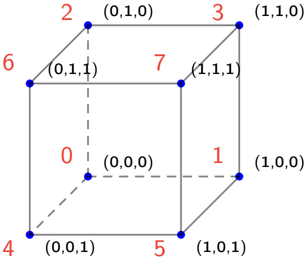


3-edges	sets
$0 - 4$	$\{3\}$
$1 - 5$	$\{3\}$
$2 - 6$	$\{3\}$
$3 - 7$	$\{3\}$

$$\text{rk}_{B^L}(0) = 3, \;\; \text{rk}_{B^L}(1) = 2, \;\; \text{rk}_{B^L}(3) = 1.$$

Logarithmic binarization

$$B^L = \{(x, y) \in \mathbb{R} \times [0, 1]^d : x = \sum_{i=1}^{d} 2^{i-1} y_i\}.$$



3-edges	sets
$0-4$	$\{3\}$
$1-5$	$\{3\}$
$2-6$	$\{3\}$
$3-7$	$\{3\}$

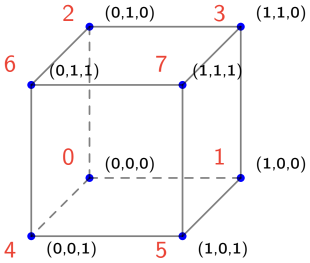
Lemma

$$\mathrm{rk}_{B^L}(\alpha) = d - f(\alpha).$$

where $f(\alpha)$ is the largest t such that 2^t divides $\alpha + 1$.

Logarithmic binarization

$$B^L = \{(x, y) \in \mathbb{R} \times [0,1]^d : x = \sum_{i=1}^{d} 2^{i-1} y_i\}.$$



3-edges	sets
$0-4$	$\{3\}$
$1-5$	$\{3\}$
$2-6$	$\{3\}$
$3-7$	$\{3\}$

Lemma

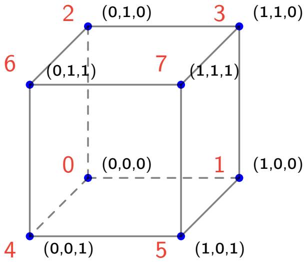
$$\mathrm{rk}_{B^L}(\alpha_1, \dots, \alpha_\ell) = d - f(\alpha_1, \dots, \alpha_\ell).$$

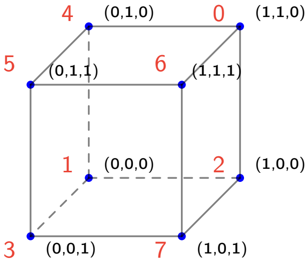
where $f(\alpha_1, \dots, \alpha_\ell) = \max\{t : 2^t \text{ divides } \alpha_j + 1 \, \forall j \in [\ell]\}$.

Hypercube binarizations

The logarithmic binarization has $\lceil \log_2(k) \rceil$ variables, but large rank.

Is there any binarization with the same number of variables, but with lower rank?





Hypercube binarizations

The logarithmic binarization has $\lceil \log_2(k) \rceil$ variables, but large rank.

Is there any binarization with the same number of variables, but with lower rank? No!

Definition

Binarization $B \subseteq [0, k] \times [0, 1]^d$ is a hypercube binarization if $\pi_y(B) = [0, 1]^d$ ($\implies d = \lceil \log_2(k) \rceil$.)

Theorem

For any hypercube binarization B,

$$\mathrm{rk}_B(\alpha_1, \dots, \alpha_\ell) \geq \mathrm{rk}_{B^L}(\alpha_1, \dots, \alpha_\ell).$$

The logarithmic binarization is optimal among hypercube binarizations.

Open questions

- What is the trade-off between the number of variables in a binarization and its rank?
- Is the unary binarization optimal among the “simplex” binarizations?
- Is there a binarization with $O(\log k)$ variables that is better than the logarithmic?

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Thank you for your attention.